Lecture (2) on classical electrodynamics I

Mohammed Shihab

Physics Department, Faculty of Science
Tanta University

Introduction

<u>Text Books:</u>

- **Sadiku, *Elements of Electromagnetics, Oxford University.***
- ***Griffiths, Introduction to Electrodynamics, Prentice Hall.***
- ***Jackson, Classical Electrodynamics,* New York: John Wiley & Sons.**
- ***Open sources: MIT open courses, Salman bin Abdelaziz Univ. ….***

<u>Evaluation will be done through:</u>

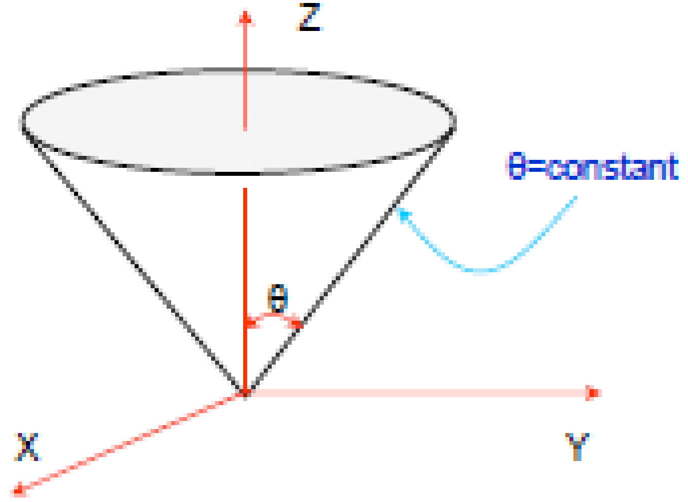
- ***<u>Final Written exam</u>***
- ***<u>Oral exam</u>***
- ***<u>Quizzes during the lecture, you need your tools (pencile, papers, calculator)</u>***
- ***<u>Homework</u>***

Keywords of the course

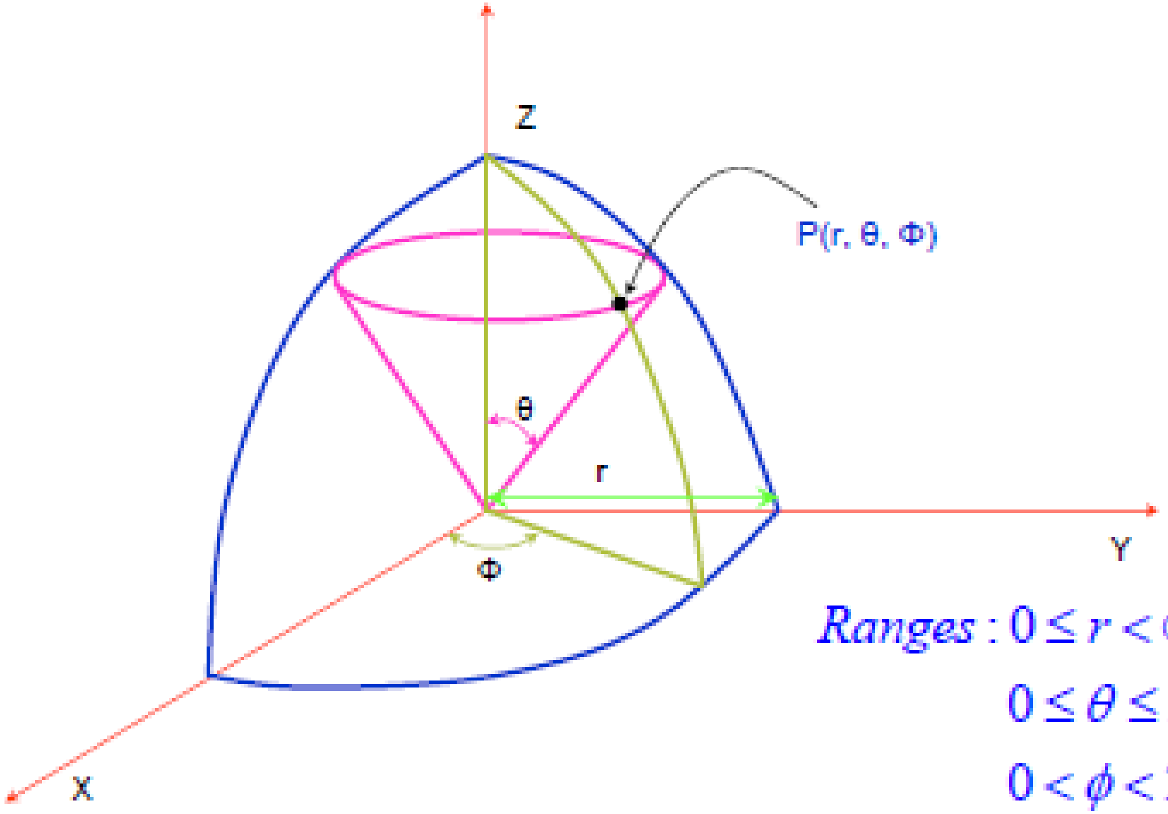
- **Vector analysis, Coordinate systems, and transformations**
- ***Electric field in materials, Polarization in dielectric materials, Continuity equation, Relaxation time, Boundary conditions***
- ***Electrostatic boundary value problems, variable separation, method of images***
- ***Magnetic fields: Bio-Savart's law and Problems, Ampere's law and problems, Analogy between electric and magnetic fields***
- ***Maxwell's equations in vacuum and matter, Problems***

Spherical Coordinates

- Any point in space is represented as the intersection of three surfaces:
  - A sphere of radius r from the origin (r=constant)
  - A cone centered around the z axis (θ=constant)
  - A vertical plane (Φ=constant)
- Any point in spherical coordinate system is considered to be at the intersection of the above three planes.

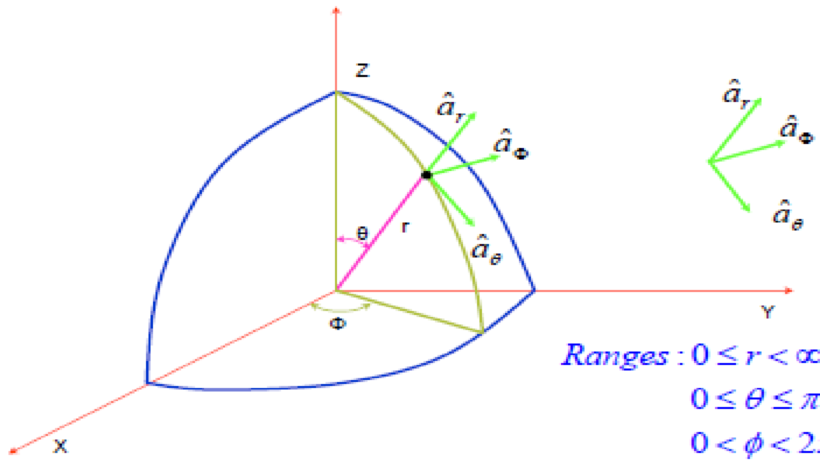


Spherical Coordinates



$$Ranges: 0 \le r < \infty$$
$$0 \le \theta \le \pi$$
$$0 < \phi < 2\pi$$

Spherical Coordinates



$$Ranges: 0 \le r < \infty$$
$$0 \le \theta \le \pi$$
$$0 < \phi < 2\pi$$

- Three unit vectors of the spherical coordinate system are shown in the figure.
- Unit vector $\hat{a}_r$ lies along the radially outward direction to the spherical surface. It lies on the cone θ=constant and the plane Φ=constant
- The unit vector $\hat{a}_\theta$ is normal to the conical surface and lies in Φ=constant plane and is tangential to the spherical surface.
- Unit vector $\hat{a}_\Phi$ is the same as in cylindrical coordinate system. It is normal to Φ=constant plane and is tangential to both the cone and the sphere.
- The unit vectors are mutually perpendicular and forms a right handed set. An RH screw when rotated from $\hat{a}_r$ *to* $\hat{a}_\theta$ will move it towards $\hat{a}_\Phi$ direction.

Spherical Coordinates

- A vector $\vec{A}$ in spherical coordinate system may be expressed as

$$\vec{A} = A_r\hat{a}_r + A_\theta\hat{a}_\theta + A_\phi\hat{a}_\phi$$

$\hat{a}_r, \hat{a}_\theta, \hat{a}_\phi$ *are unit vectors along* r, θ, ϕ *directions*

- Magnitude of the vector is given by

$$\left|\vec{A}\right| = \sqrt{A_r{}^2 + A_\theta{}^2 + A_\phi{}^2}$$

- The unit vectors $\hat{a}_r, \hat{a}_\theta, \hat{a}_\phi$ are mutually orthogonal. Thus

$$\hat{a}_r \cdot \hat{a}_r = \hat{a}_\theta \cdot \hat{a}_\theta = \hat{a}_\phi \cdot \hat{a}_\phi = 1$$

$$\hat{a}_r \cdot \hat{a}_\theta = \hat{a}_\theta \cdot \hat{a}_\phi = \hat{a}_\phi \cdot \hat{a}_r = 0$$

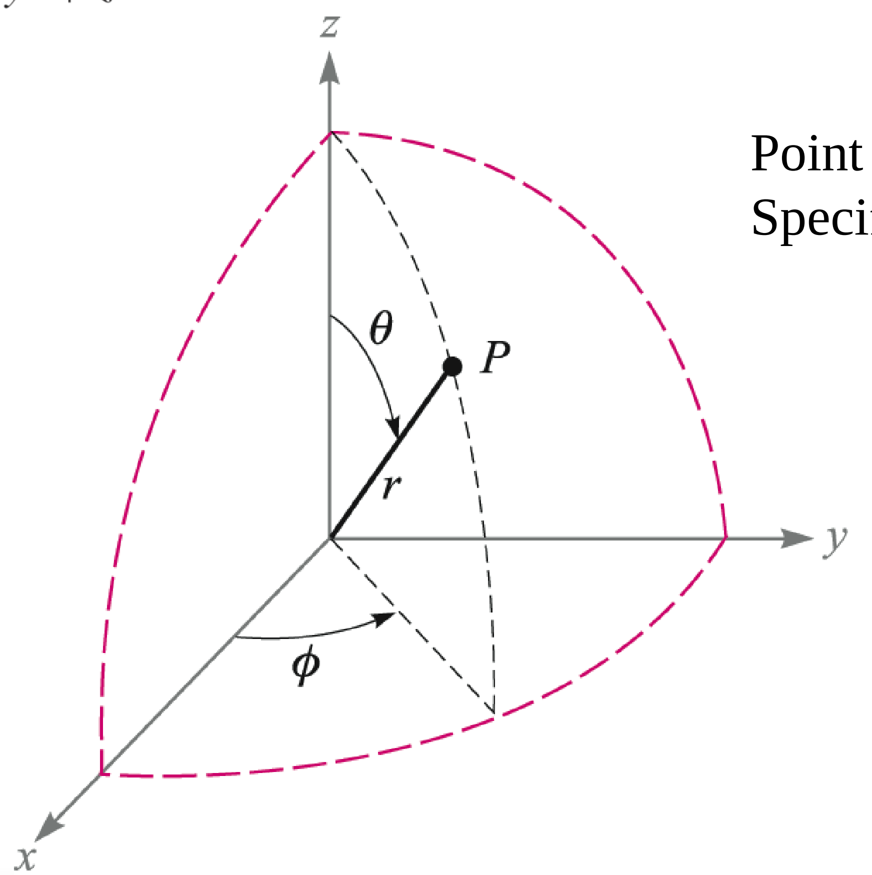
$$\boxed{\begin{aligned} \hat{a}_r \times \hat{a}_\theta &= \hat{a}_\phi \\ \hat{a}_\theta \times \hat{a}_\phi &= \hat{a}_r \\ \hat{a}_\phi \times \hat{a}_r &= \hat{a}_\theta \end{aligned}}$$

Spherical Coordinates

$$r = \sqrt{x^2 + y^2 + z^2} \qquad (r \geq 0)$$

$$\theta = \cos^{-1} \frac{z}{\sqrt{x^2 + y^2 + z^2}} \qquad (0^\circ \leq \theta \leq 180^\circ)$$

$$\phi = \tan^{-1} \frac{y}{x}$$



Point P has coordinates Specified by $P(r,\theta,\phi)$

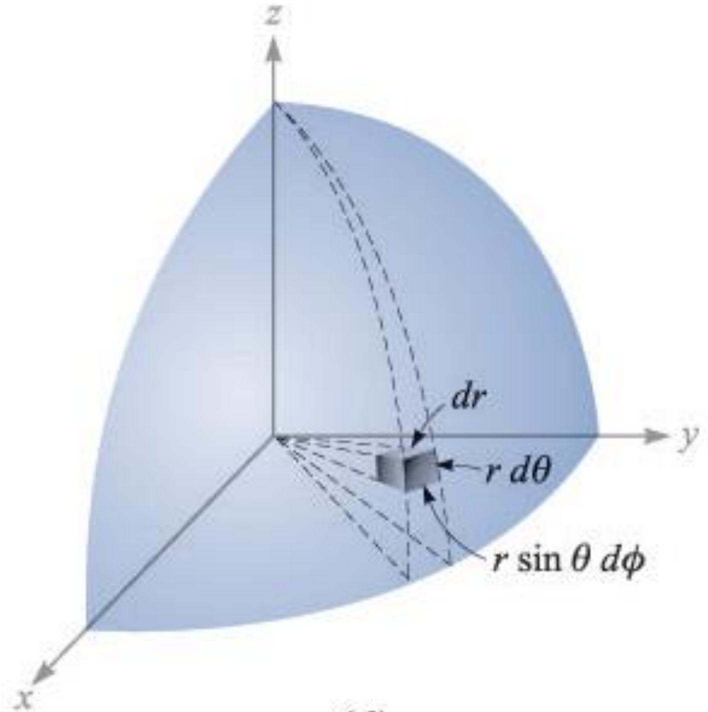
$$x = r \sin\theta \cos\phi$$

$$y = r \sin\theta \sin\phi$$

$$z = r \cos\theta$$

Differential Volume in Spherical Coordinates

$$dV = r^2 \sin\theta \; dr d\theta \, d\phi$$



(d)

Dot Products of Unit Vectors in the Spherical and Rectangular Coordinate Systems

	$\mathbf{a}_r$	$\mathbf{a}_\theta$	$\mathbf{a}_\phi$
$\mathbf{a}_x \cdot$	$\sin\theta\cos\phi$	$\cos\theta\cos\phi$	$-\sin\phi$
$\mathbf{a}_y \cdot$	$\sin\theta\sin\phi$	$\cos\theta\sin\phi$	$\cos\phi$
$\mathbf{a}_z \cdot$	$\cos\theta$	$-\sin\theta$	0

Example: Vector Component Transformation

Transform the field, $\mathbf{G} = (xz/y)\mathbf{a}_x$, into spherical coordinates and components

$$G_r = \mathbf{G}\cdot\mathbf{a}_r = \frac{xz}{y}\mathbf{a}_x\cdot\mathbf{a}_r = \frac{xz}{y}\sin\theta\cos\phi$$

$$= r\sin\theta\cos\theta\frac{\cos^2\phi}{\sin\phi}$$

$$G_\theta = \mathbf{G}\cdot\mathbf{a}_\theta = \frac{xz}{y}\mathbf{a}_x\cdot\mathbf{a}_\theta = \frac{xz}{y}\cos\theta\cos\phi$$

$$= r\cos^2\theta\frac{\cos^2\phi}{\sin\phi}$$

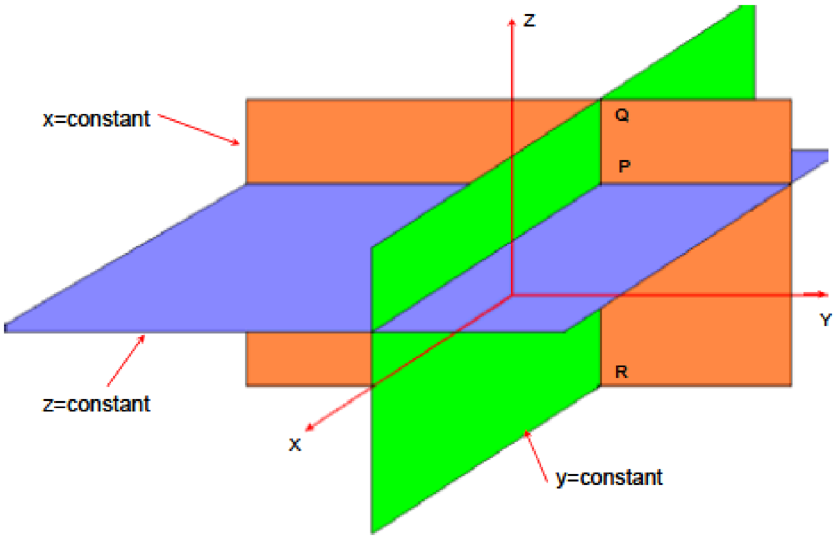
$$G\phi = \mathbf{G}\cdot\mathbf{a}_\phi = \frac{xz}{y}\mathbf{a}_x\cdot\mathbf{a}_\phi = \frac{xz}{y}(-\sin\phi)$$

$$= -r\cos\theta\cos\phi$$

$$\mathbf{G} = r\cos\theta\cos\phi\,(\sin\theta\cot\phi\,\mathbf{a}_r + \cos\theta\cot\phi\,\mathbf{a}_\theta - \mathbf{a}_\phi)$$

Constant coordinate surfaces- Cartesian system

- If we keep one of the coordinate variables constant and allow the other two to vary, constant coordinate surfaces are generated in rectangular, cylindrical and spherical coordinate systems.



- We can have infinite planes:

X=constant,

Y=constant,

Z=constant

- These surfaces are perpendicular to x, y and z axes respectively.

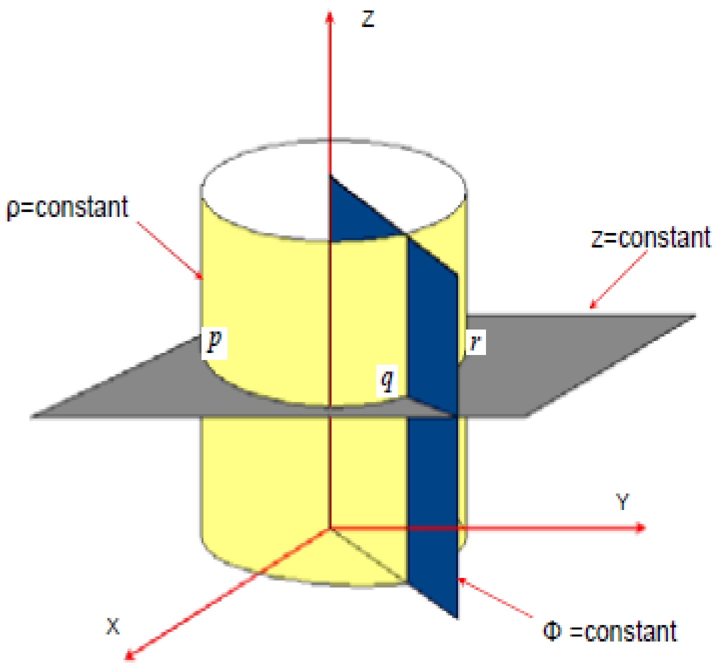
Constant coordinate surfaces-cylindrical system

- Orthogonal surfaces in cylindrical coordinate system can be generated as

  ρ=constnt

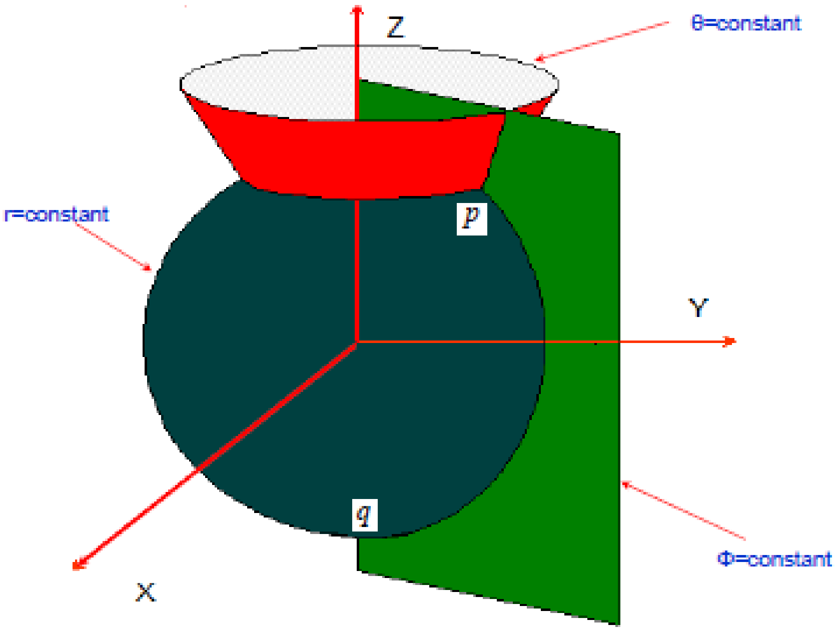
  Φ=constant

  z=constant
- ρ=constant is a circular cylinder,
- Φ=constant is a semi infinite plane with its edge along z axis
- z=constant is an infinite plane as in the rectangular system.



Constant coordinate surfaces- Spherical system

- Orthogonal surfaces in spherical coordinate system can be generated as
  r=constant
  θ=constant
  Φ=constant



- r=constant is a sphere with its centre at the origin,
- θ =constant is a circular cone with z axis as its axis and origin at the vertex,
- Φ =constant is a semi infinite plane as in the cylindrical system.

Differential elements in rectangular coordinate systems

- Differential displacement is given by

$$\overrightarrow{dl} = dx\hat{a}_x + dy\hat{a}_y + dz\hat{a}_z$$

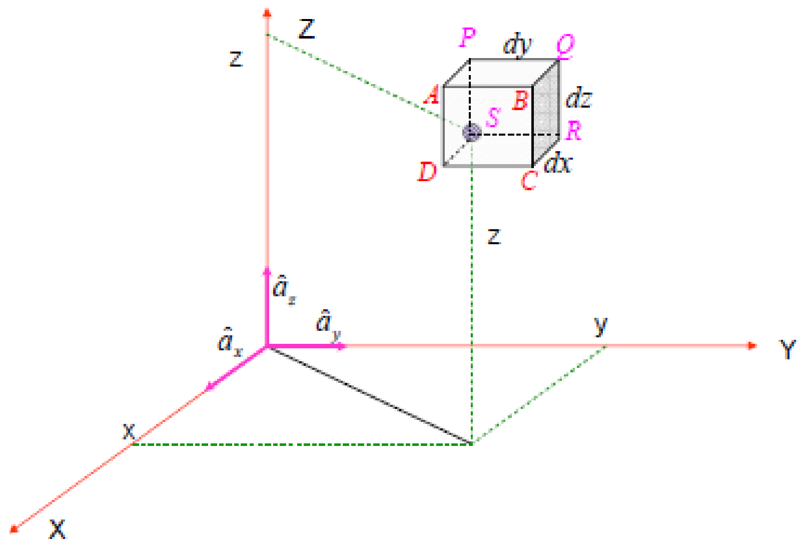
- Differential normal area is given by

$$\overrightarrow{dS} = dydz\hat{a}_x = dxdz\hat{a}_y \quad = dzdy\hat{a}_z$$

- Differential volume is given by

$$dv = dxdydz$$

- $\overrightarrow{dl}$ and $\overrightarrow{dS}$ are vectors where as dv is a scalar.



Differential elements in Cylindrical coordinate systems

- Differential displacement is given by

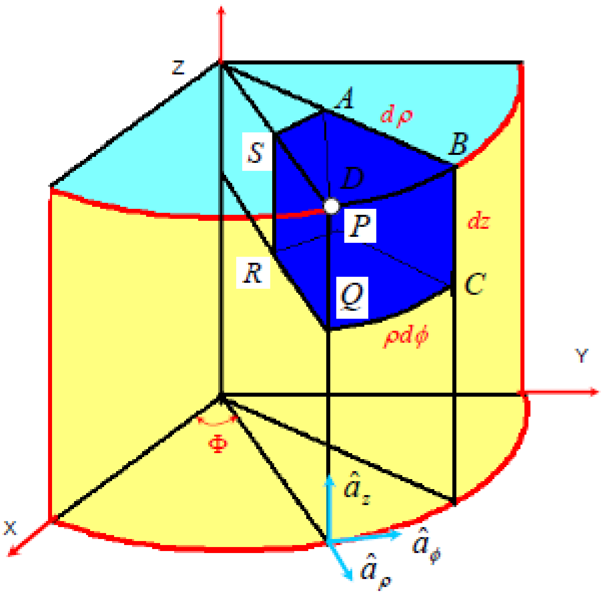
$$\overrightarrow{dl} = d\rho \hat{a}_\rho + \rho d\phi \hat{a}_\phi + dz \hat{a}_z$$

- Differential normal area is given by

$$\overrightarrow{dS} = \rho d\phi dz \hat{a}_\rho \quad = d\rho dz \hat{a}_\phi \quad = \rho d\phi d\rho \hat{a}_z$$

- Differential volume is given by

$$dv = \rho d\rho d\phi dz$$



Differential elements in Spherical coordinate systems

- Differential displacement is given by

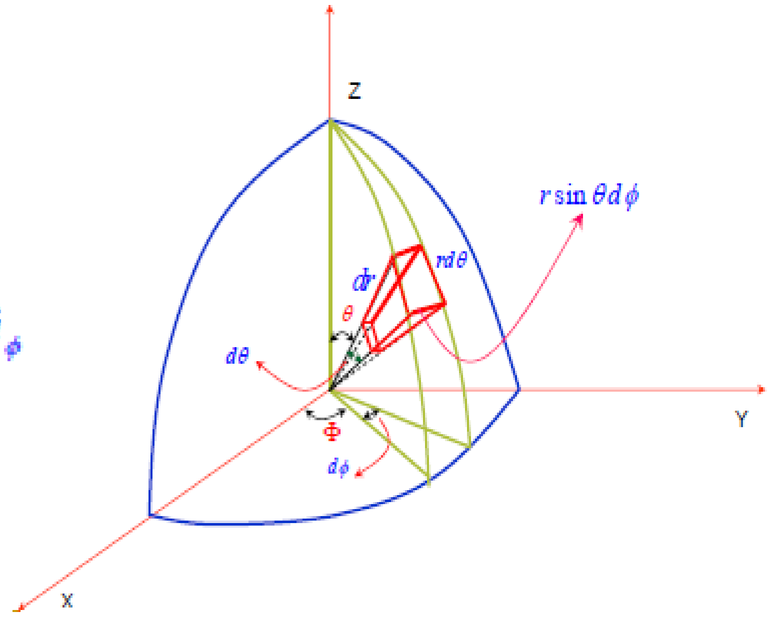
$$\overrightarrow{dl} = dr\hat{a}_r + rd\theta\hat{a}_\theta + r\sin\theta d\phi\hat{a}_\phi$$

- Differential normal area is given by

$$\overrightarrow{dS} = r^2\sin\theta d\theta d\phi\hat{a}_r = r\sin\theta dr d\phi\hat{a}_\theta = rdrd\theta\hat{a}_\phi$$

- Differential volume is given by

$$dv = r^2\sin\theta dr d\theta d\phi$$



Example (1) page 9

- **Given point P(-2,6,3) and vector** $\mathbf{A} = y\,\mathbf{a}_x + (x+z)\mathbf{a}_y$ **, express P and A in spherical coordinates. Evaluate A at P in the Cartesian and spherical coordinates.**

Example (1) page 9

Homework

- **Exercise 2.1 page 11:**

- **Time to get full mark of the exercise is next week. (*otherwise you will get half of the points*)**

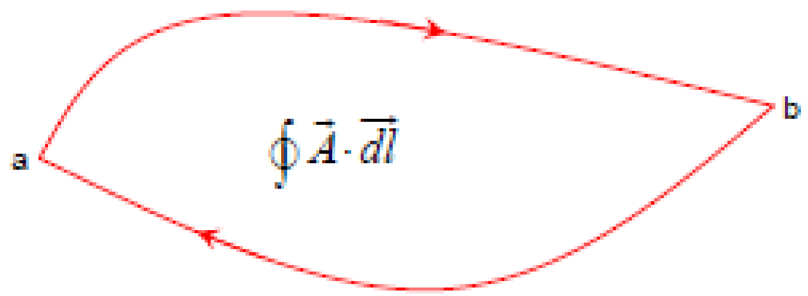
Further reading

Revise electromagnetism course I & II

Line integrals

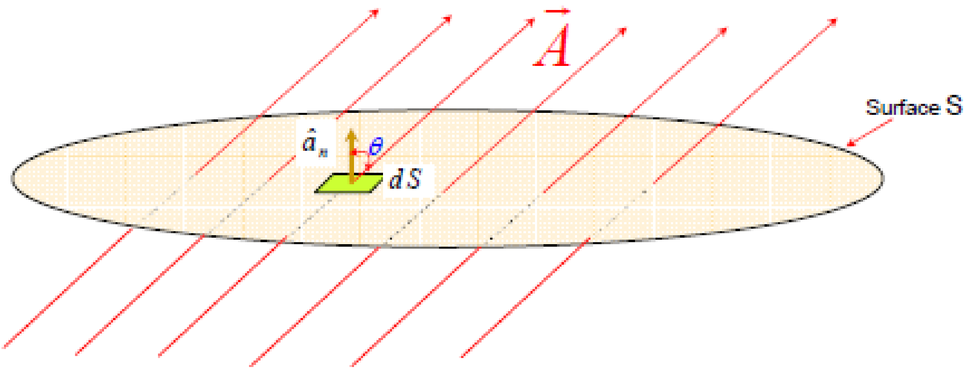
- Line integral is defined as any integral that is to be evaluated along a line. A line indicates a path along a curve in space.

  - As an example if $\vec{F}$ represents the force acting on a moving particle along a curve *ab*, then the line integral of $\vec{F}$ over the path described by the particle represents the work done by the force in moving the particle from a to b.
  - The line integral around a closed curve is called closed line integral



Surface integrals

- Consider a vector field $\vec{A}$ continuous in a region of space containing a smooth surface S.
- The surface integral of $\vec{A}$ through S can be defined as $\psi = \int_s \vec{A} \cdot \overline{dS}$



- For a closed surface defining a volume the surface integral becomes closed surface integral and is denoted by

$$\psi = \oint_S \vec{A} \cdot \overline{dS}$$

- It represents the net outward flow of flux from surface S

Volume integrals

- Let V be a volume bounded by the surface S. Let $\varphi(x,y,z)$ be a function of position defined over V. If the volume V is subdivided in to n elements of volumes $dV_1, dV_2, dV_3, \ldots\ldots, dV_n$
- In each part let us choose an arbitrary point $\varphi(x_i, y_i, z_i)$
- Then the limit of the sum $\sum \varphi(x_i, y_i, z_i) dV_i$ as $n \to \infty$ *and* $dV_i \to 0$ is called the volume integral of $\varphi(x,y,z)$ over V and is denoted by

$$\int_v \varphi \, dv$$

DEL Operator

- The del operator is the vector differential operator and is denoted by $\vec{\nabla}$. In Cartesian coordinates

$$\vec{\nabla} = \frac{\partial}{\partial x}\hat{a}_x + \frac{\partial}{\partial y}\hat{a}_y + \frac{\partial}{\partial z}\hat{a}_z$$

- DEL Operator in cylindrical coordinates:

$$\vec{\nabla} = \frac{\partial}{\partial \rho}\hat{a}_\rho + \frac{1}{\rho}\frac{\partial}{\partial \phi}\hat{a}_\phi + \frac{\partial}{\partial z}\hat{a}_z$$

- DEL Operator in spherical coordinates:

$$\vec{\nabla} = \frac{\partial}{\partial r}\hat{a}_r + \frac{1}{r}\frac{\partial}{\partial \theta}\hat{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\hat{a}_\phi$$

Gradient of a scalar field

- The gradient of a scalar field ***V*** is a vector that represents the magnitude and direction of the maximum space rate of increase of ***V***.

✓ **For Cartesian Coordinates**

$$grad\ \mathrm{V} = \vec{\nabla}\mathrm{V} = \frac{\partial V}{\partial x}\hat{a}_x + \frac{\partial V}{\partial y}\hat{a}_y + \frac{\partial V}{\partial z}\hat{a}_z$$

✓ **For Cylindrical Coordinates**

$$grad\ \mathrm{V} = \vec{\nabla}\mathrm{V} = \frac{\partial V}{\partial \rho}\hat{a}_\rho + \frac{1}{\rho}\frac{\partial V}{\partial \phi}\hat{a}_\phi + \frac{\partial V}{\partial z}\hat{a}_z$$

✓ **For Spherical Coordinates**

$$grad\ \mathrm{V} = \vec{\nabla}\mathrm{V} = \frac{\partial V}{\partial r}\hat{a}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\hat{a}_\phi$$

Divergence of a vector

- ✓ **In Cartesian Coordinates:**

$$\vec{\nabla}\cdot\vec{A}=\left(\frac{\partial A_x}{\partial x}+\frac{\partial A_y}{\partial y}+\frac{\partial A_z}{\partial z}\right)$$

- ✓ **In Cylindrical Coordinates:**

$$\vec{\nabla}\cdot\vec{A}=\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho A_\rho\right)+\frac{1}{\rho}\frac{\partial A_\phi}{\partial\phi}+\frac{\partial A_z}{\partial z}\right)$$

- ✓ **In Spherical Coordinates:**

$$\vec{\nabla}\cdot\vec{A}=\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2A_r\right)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(A_\theta\sin\theta\right)+\frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}\right)$$

Gauss's Divergence theorem

- The divergence of a vector quantity $\vec{A}$ at a given point P is the outward flux per unit volume over a closed incremental surface as the volume shrinks about P.

$$div\vec{A} = \vec{\nabla} \cdot \vec{A} = \lim_{\delta v \to 0} \frac{\oint_S \vec{A} \cdot \overrightarrow{dS}}{\Delta v}$$

$\oint_S \vec{A} \cdot \overrightarrow{dS}$ *is the net outflow of flux of a vector field* $\vec{A}$ *from a closed surface S*

- The integral of the normal component of any vector field over a closed surface is equal to the integral of the divergence of this vector field throughout the volume enclosed by the closed surface.

- The total outward flux of a vector field $\vec{A}$ through the closed surface S is the same as the volume integral of the divergence of $\vec{A}$

$$\oint_S \vec{A} \cdot \overrightarrow{dS} = \int_V \vec{\nabla} \cdot \vec{A} dV$$

Curl of a vector

- *The curl of a vector $\vec{A}$ is an axial or rotational vector whose magnitude is the maximum circulation (closed line integral) of $\vec{A}$ per unit area as the area tends to zero and whose direction is the normal direction of the area when the area is oriented so as to make the circulation maximum.*
- The circulation of a vector field $\vec{A}$ around a closed path L is the integral $\oint_L \vec{A} \cdot \overline{dL}$

$$Curl\ \vec{A} = \vec{\nabla} \times \vec{A} = \left(\lim_{\Delta S \to 0} \frac{\oint_L \vec{A} \cdot \overline{dl}}{\Delta S} \right)_{MAX} \hat{a}_n$$

Curl of a vector

- ✓ **In Cartesian Coordinates:**

$$\vec{\nabla}\times\vec{A}=\left(\frac{\partial A_z}{\partial y}-\frac{\partial A_y}{\partial z}\right)\hat{a}_x+\left(\frac{\partial A_x}{\partial z}-\frac{\partial A_z}{\partial x}\right)\hat{a}_y+\left(\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y}\right)\hat{a}_z$$

$$\vec{\nabla}\times\vec{A}=\begin{vmatrix}\hat{a}_x & \hat{a}_y & \hat{a}_z\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ A_x & A_y & A_z\end{vmatrix}$$

- ✓ **In Cylindrical Coordinates:**

$$\vec{\nabla}\times\vec{A}=\begin{vmatrix}\hat{a}_\rho & \rho\hat{a}_\phi & \hat{a}_z\\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z}\\ A_\rho & \rho A_\phi & A_z\end{vmatrix}$$

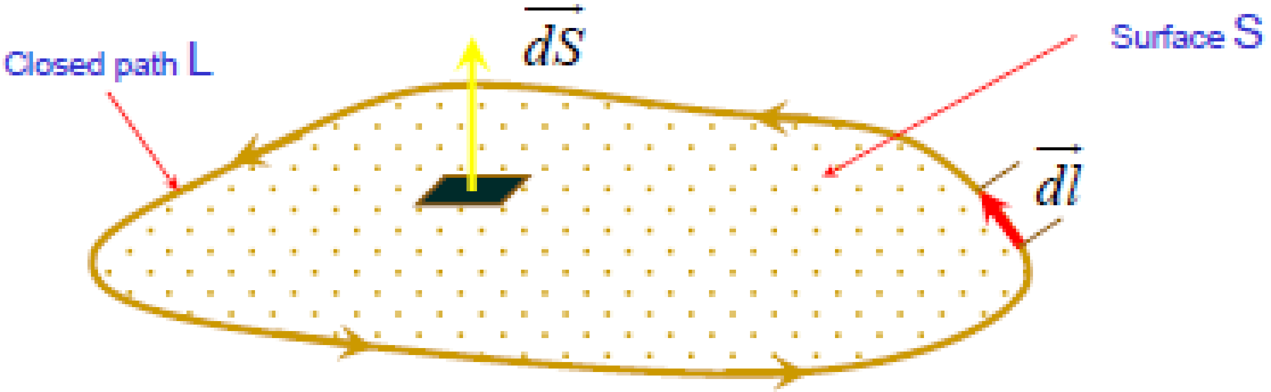
- ✓ **In Spherical Coordinates:**

$$\vec{\nabla}\times\vec{A}=\begin{vmatrix}\hat{a}_r & r\hat{a}_\theta & r\sin\theta\,\hat{a}_\phi\\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi}\\ A_r & rA_\theta & r\sin\theta\, A_\phi\end{vmatrix}$$

Stoke's theorem

- Stokes theorem states that the circulation of a vector field $\vec{A}$ around a closed path L is equal to the surface integral of the curl of $\vec{A}$ over the open surface S bounded by L provided that $\vec{A}$ and $\vec{\nabla}\times\vec{A}$ are continuous on S

$$\boxed{\oint_L \vec{A}\cdot\overrightarrow{dl} = \int_S (\vec{\nabla}\times\vec{A})\cdot\overrightarrow{dS}}$$



Laplacian of a scalar

- *The Laplacian of a scalar field V, written as $\nabla^2 V$ is the divergence of the gradient of V. It is another scalar field*

In Cartesian coordinates,Laplacian V=$\vec{\nabla}\cdot\vec{\nabla}V=\nabla^2 V$

$$=\left(\frac{\partial}{\partial x}\hat{a}_x+\frac{\partial}{\partial y}\hat{a}_y+\frac{\partial}{\partial z}\hat{a}_z\right)\cdot\left(\frac{\partial V}{\partial x}\hat{a}_x+\frac{\partial V}{\partial y}\hat{a}_y+\frac{\partial V}{\partial z}\hat{a}_z\right)$$

$$\boxed{\nabla^2 V=\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}}$$

- A scalar field V is said to be harmonic in a given region if its Laplacian vanishes in that region.

$$\boxed{\nabla^2 V=0} \Rightarrow \text{Laplace's Equation}$$

Laplacian of a scalar

In cylindrical coordinates,

$$\boxed{\nabla^2 V = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial V}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2}}$$

In spherical coordinates,

$$\boxed{\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 V}{\partial \phi^2}}$$

Laplacian of a vector $\vec{A}$ denoted as $\nabla^2 \vec{A}$ is defined as

$$\boxed{\nabla^2 \vec{A} = \vec{\nabla}\left(\vec{\nabla} \cdot \vec{A}\right) - \left(\vec{\nabla} \times \vec{\nabla} \times \vec{A}\right)}$$