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TWO SEMICLASSIFYING THEOREMS FOR METABELIAN GROUPS

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ABSTRACT

In this paper, a certain description will be given for finitely generated groups with a cyclic factor group by an abelian subgroup and for finite p-groups with a cyclic commutant. We can not expect to get a full classification for metabelian groups, because the problem of the classification of p-groups with a cyclic factor group of order p^2 by an abelian subgroup, and the problem of the classification of finite p-groups with a cyclic commutant of order p^2 are reduced to the classical unsolved problem on the classification of pairs of linear pperetors in the finite dimensional vector space (see [7]). The classification of finite p-groups with an abelian subgroup of index p is given in [10] (see also [5-6]), and the classification of finitely generated groups with a commutant of prime order is given in [8]

1- FINITELY GENERATED GROUPS WITH A FINITE CYCLIC FACTOR GROUP BY AN ABELIAN SUBGROUP.

We will refer to the group homomorphism $\Phi : G \longrightarrow H$ as a quasiisomorphism, if its kernel and image index $H/\mathrm{Im}\,\Phi$ are finite.

Note [2], that the indecomposable integer polynomial

$$f_d(x) = \prod_w (x-w),$$

where the product is taken for all primitive $d^{\underline{th}}$ roots of unity, is said to be a $d^{\underline{th}}$ circular polynomial. The degree of the polynomial $f_d(x)$ is equal to $\Phi(d)$, where Φ is the Euler's function,

$$x^m - 1 = \prod_{d \mid m} f_d(x).$$

Let us fix a natural number m and for each circular polynomial $f_d(x)$, $d|m$, define a group G_d with generators g,a and defining relations

$$g^m = 1,\ a^{f_d(g)} = 1,\ [a^{g^i}, a^{g^j}] = 1\ (0 \leqslant i < j < \deg(f_d)),$$

where

$$a^{\alpha_0 + \alpha_1 g + \dots + \alpha_k g^k} = a^{\alpha_0} \cdot g^{-1} a^{\alpha_1} g \dots g^{-k} a^{\alpha_k} g^k,\ \alpha_i \in Z.$$

THEOREM 1. Each finitely generated group with a finite cyclic factor group, of order m, by an abelian subgroup is quasiisomorphic to a uniquely determined direct product

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$$G_{d_1} \times \dots \times G_{d_n} ,$$

where $d_1 \leqslant \dots \leqslant d_n$ are the divisors of m.

The idea of this theorem and the method of its proof are prompted from the classification, up to quasiisomorphism, of finitely generated modules over the ging Z[[x]] of formal power series with integer p-aidic coefficients, obtained by Z.P. Serr and P. Con [3], A quasiisomorphism of modules $\Phi: A \longrightarrow B$ is a homomorphism Φ for which $\ker(\Phi)$ and $B/\mathrm{Im}(\Phi)$ are finite.

The following lemma is necessary for the proof of t theorem 1.

LEMMA 1. Each finitely generated module A over the ring $Z[x]/(x^m - 1)$ is quasiisomorphic to the uniquely determined direct sum

$$Z[x]/(f_{d_1}) \oplus \dots \oplus Z[x]/(f_{d_n}), \qquad (1)$$

where $f_{d_i}(x)$ are circular polynomials and $d_1 \leqslant \dots \leqslant d_n$ are the divisors of m.

PROOF. The number of the defining relations for the module A is finite. Let

$$\lambda_{i1} U_1 + \dots + \lambda_{in} U_n = 0, \quad 1 \leq i \leq r,$$

be the defining relations for A. The elements of the relations matrix $M = (\lambda_{ij})$ are integer polynomials determined modulo $x^m - 1$. Reselecting generators $u_1, \dots, u_n$ we can carry out ordinary elementary transformations on the columns of the matrix M: rearrange columns and add to one column another one, multiplied by an integer polynomial. The same transformations can be carried out on the rows of M.

We point out two more transformations which correspond to the transition to the quasiisomorphic module. The row $(z\lambda_1, \dots, z\lambda_n)$, $0 \neq z \in Z$, could be divided by z ; this corresponds to the transition to the factor module by a finite submodule. The i-th column could be multiplied by $\mu \in Z[x]$, $(\mu, x^m - 1) = 1$; this corresponds to the transition to a suppermodule $A' \supset A$, which is obtained by joining a new generator u'_i and a relation $\mu u'_i = u_i$. Since $\mu\phi + (x^m - 1)\Psi = z$ for some $\phi, \Psi \in Z[x]$, $0 \neq z \in Z$, then $zu'_i \in A$

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and the factor module A'/A is finite.

We show, by such transformation, that the matrix M can be reduced to the diagonal form. By admissible transformations we make λ_{11} a non-zero polynomial of an extremely small degree. Suppose
$d = (\lambda_{11}, x^m - 1) \in Z[x]$. Then

$$zd = \lambda_{11}\phi + (x^m - 1)\psi$$

for some $\phi, \psi \in Z[x]$, $0 \neq z \in Z$. Since $(\phi, x^m - 1) = 1$, then the first column could be multiplied by ϕ and we get a new element $\lambda_{11} = zd \pmod{x^m - 1}$. Having multiplied all columns, except the first one, by z and dividing the first row by z, we get a new $\lambda_{11} = d$.

Let $\lambda_{1i} \equiv 0 \ (\mathrm{mod}(x^m - 1)$, $i>1$. Divide λ_{1i} by d: $\lambda_{1i} = d\phi + r$. Since $d \mid x^m - 1$. then the coefficient of the higher term in d is equal to 1, and hence ϕ, $r \in Z[x]$. Subtracting from the i - th column the first, multiplied by ϕ, we get a new $\lambda_{1i} = r$. Since $\deg(r) < \deg(d)$, then $r=0$, otherwise the degree of λ_{11} would not be extremely small.

Analogously, we replace all the elements λ_{i1}, $i>1$, by zeros, and obtain

$$M = \begin{pmatrix} d & 0 \ldots 0 \\ 0 & \\ \vdots & N \\ 0 & \end{pmatrix}$$

Having repeated the reduction enough times, we reduce M to the diagonal form, where the diagonal elements are divisors of the polynomial $x^m - 1$.

If a diagonal element has the form fg, where $(f,g) = 1$, then it can be replaced by two diagonal elements f and g. This can be demonstrated by the following chain of transformations on the relation matrix ($z = f\phi + g\psi$, $0 \neq z \in Z, \phi, \psi \in Z[x]$):

$$(fg) \to \begin{pmatrix} 1 & 0 \\ 0 & fg \end{pmatrix} \to \begin{pmatrix} 1 & f \\ 0 & fg \end{pmatrix} \to \begin{pmatrix} 1 & f \\ -g & 0 \end{pmatrix} \to \begin{pmatrix} f & 1 \\ 0 & g \end{pmatrix} \to$$

$$\to \begin{pmatrix} f & z \\ 0 & zg \end{pmatrix} \to \begin{pmatrix} f & f\phi+g\psi \\ 0 & g \end{pmatrix} \to \begin{pmatrix} f & g\psi \\ 0 & g \end{pmatrix} \to \begin{pmatrix} f & 0 \\ 0 & g \end{pmatrix}.$$

Consequently, the diagonal elements of the matrix M could be

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made circular polynomials, and the lemma is proved.

PROOF OF THEOREM 1. Let G be a finitely generated group with a cyclic factor group by an abelian subgroup $G/A = \langle \omega \rangle$ of order m, and suppose $g \in \omega$. We can consider $g^m = 1$, otherwise we change the group G by a quasiisomorphic group obtained by joining to the center of the group G an element a, such that $a^m = g^m$ and we get $(ga^{-1})^m = 1$.

The conjugation action of the element g on the normal subgroup A transfers it into a module over the integer group ring $Z \langle g \rangle \simeq Z[x]/(x^m - 1)$. Defining G up to quasiisomorphism, we can decompose the module A into the direct sum (1) and, consequently, identify G with a subgroup of finite index in $G_{d_1} \times \ldots \times G_{d_n}$.

2- FINITE P-GROUPS WITH A CYCLIC COMMUTANT.

We will say that the group G is decomposed into a commutative product of its subgroups $A_1, \ldots, A_n$ if they generate G and $[a_i, a_j] = 1$ for all $a_i \in A_i$, $a_j \in A_j$, $i \neq j$. Obviously, G ia a factor group of the direct product $A_1 \times \ldots \times A_n$. By Z(G) we denote the center of the group G.

THEOREM 2. If G is a finite p-group with a cyclic commutatnt G', then G itself or (in the scase $p = 2 = |G' \cap Z(G)| < |G'|$) some subgroup of index 2 is decomposable into a commutative product of subgroup with a maximum of two generators.

The classification of p-groups with two generators and a cyclic commutant is given in [4]. We need the following lemma for the proof of the theorem. This lemma is a generalization of the known lemma of V.A. Sheriev [9].

LEMMA 2. Let G be a finite p-group, and $H<G$ be a subgroup with two generators and cyclic commutant $H' = [H,G]$ such that $|H'| = 2$ when $|H' \cap Z(H)| = 2$. Then $G = H.\ C(H)$, where C(H) is the centeralizer of H in G.

PROOF, Let $H = \langle h,k \rangle$, $g \in G$. It is sufficient to show that there exists a number α such that

$$[\ k,h^{\alpha} g] = 1. \qquad (2)$$

Hence, analogously, there exists a number β such that $[\ h,k^{\beta}h^{\alpha} g] = 1$, i.e. $K^{\beta} h^{\alpha} g \in C(H)$, $g \in H.C(H)$, which proves our lemma.

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Suppose $[h,k] = a$. Since $H' = [H,C] = \langle a \rangle$, then $[k,g^{-1}] = a^s$ for some s. Condition (2) can be rewritten in the form $k^{-1}g^{-1}k\,[k,h^{\alpha}]g = 1$, i.e. $[k,h^{\alpha}] = [k,g^{-1}]$, and consequently

$$[\,k,h^{\alpha}\,] = a^s . \qquad (3)$$

Let $h^{-1}ah = a^t$, $|a| = p^{\lambda}$. Then

$$[\,k,h^{\alpha}\,] = [\,k,h\,] \cdot h^{-1}[\,k,h\,]\,h \;\ldots\; h^{-\alpha+1}[\,k,h\,]\,h^{\alpha-1}$$

$$= a^{1+t+\ldots+t^{\alpha-1}}.$$

Hence, condition (3) is rewritten in the form

$$1 + t + \ldots + t^{\alpha-1} = s(\mathrm{mod}\ p^{\lambda}). \qquad (4)$$

Since $h^{-p^{\mu}}\,a\,h^{p^{\mu}} = a$, where $p^{\mu} = |h|$, then

$$t^{p^{\mu}} = 1(\mathrm{mod}\ p^{\lambda}),\ t \equiv 1(\mathrm{mod}\ p).$$

If $4 \mid p^{\lambda}$, then, from the coditions of the lemma, $|H' \cap Z(H)| > 4$ and therefore

$$a^{2^{\lambda-2}} = h^{-1}a^{2^{\lambda-2}}h = a^{t2^{\lambda-2}},\ t \equiv 1(\mathrm{mod}\ 4).$$

Consequently, $t = 1 + m\,p^{\nu}$, where $p \nmid m$, $\nu \geqslant 1$; $\nu > 1$ when $4 \mid p^{\lambda}$.

Multiplying condition (4) by $t - 1 = mp^{\nu}$, we rewrite it in the form

$$t^{\alpha} = 1 + smp^{\nu} \pmod{p^{\lambda+\nu}}. \qquad (5)$$

The element $t = 1 = mp^{\nu}$ in the multiplicative group of the ring $Z/p^{\lambda+\nu}Z$ generates the subgroup of all elements of the form $1 + zp^{\nu}$, $z \in Z$ [1], and so there exists for which condition (5) if satisfied, and consequently, condition (2).

PROOF OF THEOREM 2. Suppose that the condition

$$p = 2 = |G' \cap Z(G)| < |G'| \qquad (6)$$

is not satisfied, and let $g_1, \dots, g_n$ be some system of generators for the group G. We can consider that the element $[g_1, g_2]$ generates the commutant G', By virtue of lemma 2, on multiplying by suitable powers of g_1 and g_2, the elements $g_3, \dots, g_n$ are commutative with g_1 , g_2 and we get a commutative product

$$G = \langle g_1, g_2 \rangle \langle g_3, \dots, g_n \rangle.$$

By induction, we obtain a commutative product

$$G = \langle g_1 , g_2 \rangle\langle g_3 , g_4 \rangle \dots \langle g_{2k-1}, g_{2k} \rangle\langle g_{2k+1} \rangle\dots\langle g_n\rangle.$$

Now, suppose that condition (6) is satisfied. Then,in the system of generators $g_1, \ldots, g_n$ for the group G there exists an element, let it be g_1, which doesn't commute with the element $b \in G'$, $|b| = 4$. Hence $g_1^{-1} b g_1 = b^{-1}$. Consider, in G, the subgroup $N = \langle g_1^2, g_2', \ldots, g_n', a \rangle$ of index 2, where a is the generator of the commutant G', $g_i' = g_i$ if $g_i^{-1} b g_i = b$, while $g_i' = g_i g_1$ if $g_i^{-1} b g_i = b^{-1}$. Obviously, $b \in Z(N)$, and therefore either $|N'| = 2$ or $|N' \cap Z(N)| > 2$. Consequently, the arguements of the previous paragraph are applicable to N. The theorem is proved.

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نظريتان شبه مصنفتان للزمر الميتابليه

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يعطى فى هذا البحث وصف للزمر G نهائية التوليد المشتملة على زمرة حاصل قسمة G/A دائرية حيث A زمرة جزئية آبيليه ، كما يعطى وصف للزمر G التى رتبتها عدد اولى p وزمرتها الجزئية المتولده بجميع العناصر [a,b] دائرية حيث G a,b : هذا الوصف لهذين النوعين من الزمر يؤدى الى تقسيم كل منهما الى فصول منفصلة بحيث تكون كل زمرتين منتميتين الى فصل واحد شبه ايزومورفيتين (quasiisomorphic) .