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ON A SINGULAR DIFFERENTIAL OPERATOR OF THE SECOND ORDER

BY

A.A. Darwish, M.S. Selem, E.E. Abd El-Aal
Mathematics Department, Faculty of Science,
Tanta University

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ABSTRACT

In this paper we are concerned with a differential operator of the second order with discontinuous coefficient. The continuous spectrum and the resolvent set of the operator are investigated and the spectrum of the adjoint operator is obtained.

INTRODUCTION

In the space $\mathcal{L}_2(o\ ,\infty)$ we consider the singular operator L, generated by the differential expression

$$\ell(y) = \frac{1}{\rho\ (x)} \left\{ -y'' + q(x)y \right\}, \qquad o \leqq x < \infty,$$

and associated with the condition $y'(o) = o$. The function $q(x)$ is a complex valued, continuous function on $[o\ ,\infty)$ and $\int_o^\infty x \left| q(x) \right| dx < \infty$. (I)

Also, the coefficient $\rho(x)$ is a discontinuous function at x = 1 :

$$\rho(x) = \begin{cases} a^2 , & o \leqq x \leqq 1 \\ 1 , & 1 < x < \infty , \; a \neq 1, \; \mathrm{Im}\, a \neq o. \end{cases}$$

It should be mentioned that the spectrum of the operator L has been studied in many works. Naimark [4] has considered the same differential operator when $\rho(x) = 1$ on half line with a condition at o. Krall [2] studied the operator when $\rho(x) = 1$ on $[o , \infty)$ with an integral boundary condition. Petrosian [3] investigated the operator with the condition $y(o) = o$. Darwish considered in [1] the case when $\rho(x)$ is a real discontinuous function with integral boundary condition.

1. The operator L :

We denote by D(L) those functions y defined on $[o , \infty)$ and satisfying the conditions

i) y is in $\mathcal{L}_2(o , \infty)$;

ii) y' exists and is absolutely continuous on every finite subinterval of $[o , \infty)$;

iii) $\ell(y)$ is in $\mathcal{L}_2(o , \infty)$;

iv) $y'(o) = o$.

If we define the operator L by $Ly = \ell(y)$ for all $y \in D(L)$ then, the eigenvalue equation is of the form

$$Ly = \lambda y$$

2. Some solutions of the equation $\ell(y) = \lambda y$:

Now, consider the differential equation

$$-y'' + q(x)y = \lambda \rho (x)y \ , \ o \leqq x < \infty . \qquad (1)$$

Let $\lambda^{\frac{1}{2}} = k = \sigma + i\tau$ such that $o \leqq \arg k \leqq \pi$. Since $q(x)$ satisfies condition (I), then denoting $F_1(x,k) = F(x,-k)$, we can use the solutions $F(x,k)$ and $F_1(x,k)$ of Naimark [2. p. 297 - 299] of equation (1) as $1 \leqq x < \infty$.

The properties of the solutions $F(x,k)$, $F_1(x,k)$:

i) Due to [5] the solutions $F(x,k)$ and $F_1(x,k)$ are jointly continuous in x,k for all $x > o, \tau \geqq o$, $k \neq o$ and holomorphic in k for all $\tau > o$. These solutions have the following asymptotic behaviour

$$F(x,k) = e^{ikx}(1 + o(1)) \ , \quad F'(x,k) = e^{ikx}(ik + o(1)) \ , \qquad (2)$$

$$F_1(x,k) = e^{-ikx}(1 + o(1)), \ F_1'(x,k) = e^{-ikx}(-ik + o(1)),$$

as $x \to \infty$ for all $\tau \geqq o$, $k \neq o$ and

$$F(x,k) = e^{ikx}(1 + O(\frac{1}{k})) \ , \ F'(x,k) = ike^{ikx}(1 + O(\frac{1}{k}))$$

$$F_1(x,k)= e^{-ikx}(1 + O(\frac{1}{k})) \ , \ F_1'(x,k) = -ike^{-ikx}(1 + O(\frac{1}{k}))$$

as $|k| \to \infty$ for all x and $\tau \geq o$.

Further

$$W[F(x,k),\ F_1(x,k)] = F(x,k)\ F_1'(x,k) - F'(x,k)\ F_1(x,k) = -2ik$$

ii) In view of [1], we can write the following expressions for the solutions $F(x,k)$ and $F_1(x,k)$ on the half-line :

$$F(x,k) = \begin{cases} e^{ik}[\cos k(x-1)a + \frac{i}{a} \sin k(x-1)a](1 + 0(\frac{1}{k})), & o \leqq x \leqq 1 \\ e^{ikx}(1 + 0\ (\frac{1}{k}))\ , & 1 \leqq x < \infty . \end{cases}$$

and

$$F_1(x,k) = \begin{cases} e^{-ik}[\cos ka(x-1) - \frac{i}{a} \sin ka(x-1)](1+0(\frac{1}{k})), & o \leqq x \leqq 1 \\ e^{-ikx}(1 + 0(\frac{1}{k})), & 1 \leqq x < \infty . \end{cases}$$

for $|k| \to \infty$ and $\tau \geqq o$.

Moreover as $x \to \infty$ and $\tau \geqq o$ we have

$$F(x,k) = \begin{cases} e^{ik}[\cos ka(x-1) + \frac{i}{a} \sin ka(x-1)](1 + o(1)), & o \leqq x \leqq 1 \\ e^{ikx}(1 + o(1))\ , & 1 \leqq x < \infty \end{cases}$$

and

$$F_1(x,k) = \begin{cases} e^{-ik}[\cos ka(x-1) - \frac{i}{a} \sin ka(x-1)](1+ o(1)), & o \leqq x \leqq 1 \\ e^{-ikx}(1 + o(1))\ , & 1 \leqq x < \infty \end{cases}$$

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Theorem 1 :

i) If λ is not an eigenvalue and $W(k) \neq o$ then, the resolvent operator is an integral operator

$$R_\lambda(\rho\phi) = \int_o^\infty R(x\ ,\ t\ ,\ k)\,\rho(t)\,\phi\,(t)\,dt$$

with the kernel

$$R(x,t,k) = \begin{cases} \dfrac{F(x,k)F_1'(o,k)F(t,k)}{2ik\ W(k)} - \dfrac{F(x,k)F_1(t,k)}{2ik}, & t \geqslant x \\[2ex] \dfrac{F(x,k)F_1'(o,k)F(t,k)}{2ik\ W(k)} - \dfrac{F(t,k)F_1(x,k)}{2ik}, & t \leqslant x \end{cases} \qquad (3)$$

where $W(k) = F'\ (o,k)$.

ii) All numbers $\lambda = k^2$, $W(k) \neq o$, $\tau > o$ belong to the resolvent set of the operator.

Proof : By the assumption $\tau > o$ and $W(k) \neq o$, so $F(x,k)$ and $F_1(x,k)$ are linearly independent solutions of (1). It follows from variation of parameters that the general solution of $\ell(y) - \lambda y = \phi$ is $y(x) = R_\lambda\,(\rho\phi) = \int_o^\infty R(x,t,k)\,\rho(t)\,\phi\,(t)dt$, where $R(x,t,k)$ is the required result and hence, (i) is proved.

Now, taking into account [1. p. 126], it can be deduce that the eigenvalues of the operator L are given by the solutions k in the upper half plane of $W(k) = o$. Hence it follows immediatly that all numbers $\lambda = k^2$, $W(k) \neq o$, $\tau > o$ belong to the resolvent set of the operator L. This completes the proof.

Theorem 2 : Every point of the semi axis $\lambda \geqq o$ is a point of the spectrum.

Proof : Let $u(t,k) = -\frac{1}{2ik} F_1'(o,k)\, F(t,k)$ then, substituting in (3) as $t < x$ we have

$$R(x,t,k) = -\frac{u(t,k)\, F(x,k)}{W(k)} - \frac{1}{2ik} F_1(t,k)\, F(x,k).$$

Now, let

$$f(x,k) \cong \begin{cases} \bar{F}_1(x,k) - \dfrac{\bar{u}(x,k) \int_0^a \bar{F}_1(t,k) u(t,k)\,dt}{\int_0^a |u(t,k)|^2\,dt} & , x \leqq a \\ 0 & , x > a \end{cases}$$

Thus,

$$\int_0^a f(x,k)u(x,k)dx = \int_0^a F_1(x,k)u(x,k)dx - \int_0^a \bar{u}(x,k)u(x,k)dx$$
$$\int_0^a \bar{F}_1(t,k)u(t,k)dt \Big/ \int_0^a |u(t,k)|^2\,dt$$

$$= \int_o^a \bar{F}_1(x,k)u(x,k)dx - \int_o^a |u(x,k)|^2 dx$$

$$\cdot \int_o^a \bar{F}_1(t,k)u(t,k)dt / \int_o^a |u(t,k)|^2 dt$$

$$= \int_o^a F_1(x,k)u(x,k)dx - \int_o^a \bar{F}_1(t,k)u(t,k)dt = o$$

and

$$\int_o^a f(x,k)\bar{f}(x,k)dx = \int_o^a f(x,k)F_1(x,k)dx - \int_o^a f(x,k)u(x,k)dx$$

$$\cdot \int_o^a F_1(t,k)\bar{u}(t,k)dt / \int_o^a |u(t,k)|^2 dt$$

$$= \int_o^a f(x,k)F_1(x,k)dx.$$

$$R_\lambda f(x,k) = \int_o^\infty \frac{-u(t,k)F(x,k)f(t,k)}{W(k)} dt - \frac{1}{2ik} \int_o^\infty F_1(t,k)F(x,k) f(t,k)dt$$

$$= \int_o^a \frac{-u(t,k)F(x,k)f(t,k)}{W(k)} dt - \frac{1}{2ik} \int_o^a F_1(t,k)F(x,k) f(t,k)dt$$

$$= \int_o^a \frac{-F(x,k)}{2ik} F_1(t,k)f(t,k)dt = \frac{-F(x,k)}{2ik} \cdot \int_o^a F_1(t,k)f(t,k)dt.$$

Hence,

$$\| R_\lambda f(x,k) \|^2 = \int_o^\infty | R_\lambda f(x,k) |^2 dx \geqq \int_a^\infty | R_\lambda f(x,k) |^2 dx$$

$$= \int_a^\infty \left| \frac{-F(x,k)}{2ik} \int_o^a | f(t,k) |^2 dt \right|^2 dx$$

$$= \frac{1}{4|k|^2} \int_a^\infty | F(x,k) |^2 dx . \int_o^a | f(t,k) |^4 dt$$

Since $F(x,k) = \exp(ikx)(1 + o(1))$ as $x \to \infty$ (formula (2)), there exists a suffeiciently large a such that for $a < x < \infty$, $\tau \geqq o$, $k \neq o$; $|F(x,k)| \geqq \frac{1}{2} \exp(-\tau x)$, (see [5]), thus

$$\int_o^\infty | F(x,k) |^2 dx \geqq \frac{1}{8\tau} \exp(-2\tau a).$$

Since in any semi circle in the upper half plane with a centre at the origin of coordinates $\int_o^a | f(x,k) | dx$ is bounded away from zero, then

$$\| R_\lambda \|^2 \geqq \frac{ce^{-2\tau a}}{4|k|^2 8\tau}, \text{ where } c \text{ is a constant.}$$

From here, it follows that $\| R_\lambda \| \to \infty$ as $\tau \to o$ and so, the square of the point is in the spectrum of the operator L and hence the theorem is proved.

Lemma 1 : The adjoint operator to the operator L is defined by the differential equation $-z'' + \bar{q}(x)z = \lambda\bar{\rho}(x)z$ and the condition $z'(o) = o$. The proof follows immediately by using [1].

Theorem 3 : The continuous spectrum of the operator L lies on the half semi axis $\lambda \geqq o$.

Proof : We have to show that the domain of the resolvent operator i.e., the range of the operator $(L - \lambda E)$ is dense in $L_2(o,\infty)$. This is equivalent to prove that the orthogonal complement of this range is the zero element. But, as in [4] the orthogonal complement coincides with the space of solutions of $L^* z = \lambda\bar{\rho}z$. From Lemma 1, it is evident that the adjoint operator L^* of L can be defined by the following differential equation $-z'' + \bar{q}(x)z = \lambda\bar{\rho}z$ with the condition $z'(o) = o$. Conversely, suppose that there is a function $z \in \mathcal{L}_2(o,\infty)$ which is different from zero and such that $(Ly,z) = (y, L^* z)$. Hence for $\lambda \geqq o$, there exists a non-trivial solution of the equation $(y, L^* z) = o$ which belongs to $\mathcal{L}_2(o,\infty)$. Since the operator L does not have positive eigenvalues (see [4]) then, there is no non trivial solution of the required equation belongs to $\mathcal{L}_2(o,\infty)$ and the assertion follows.

4. The spectrum of the operator L^* :

To investigate the spectrum of the operator L^* we resort to the following theorem [2] :

Theorem 4 : Let H be an arbitrary Hilbert space and L be a linear operator on H with adjoint L^* then :

1- If λ is in the residual spectrum of L, $\bar{\lambda}$ is in the point spectrum of L^*.

2- If λ is in the point spectrum of L, $\bar{\lambda}$ is in the point or residual spectrum of L^*.

3- If λ is in the continuous spectrum of L , $\bar{\lambda}$ is in the continuous spectrum of L^*.

4- If λ is in the resolvent set of L, $\bar{\lambda}$ is in the resolvent set of L^*.

This theorem leads to the following which can be easily proved.

Theorem 5 : The spectrum of L^* consists of :

1) eigenvalues $\bar{\lambda}$ whenever $W(k) = o$ and $\lambda = k^2$ is not on the positive axis $\lambda \geqq o$.

2) continuous spectrum is on the positive half-axis $\lambda \geqq o$.

Corollary : All numbers $\bar{\lambda}$ whenever $W(k) \neq o$ and $\lambda = k^2$, Im $k > o$ belong to the resolvent set of the operator L^*.

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عن مؤثر تفاضلى شاذ من الرتبة الثانية

عادل عبد الفتاح درويش – مؤنس صالح سليم – السيد السيد عبد العال
قسم الرياضيات – كلية العلوم – جامعة طنطا

فى هذا البحث تناولنا دراسة مؤثر تفاضلى شاذ من الرتبة الثانية له معامل غير متصل . لقد فحصنا الطيف المتصل وفئة القيم المنتظمة للمؤثر ، كما حصلنا على طيف المؤثر المرافق للمؤثر محل الدراسة .