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ON COMPACT CLOSURE SPACES

BY

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ABSTRACT

In 1981, Thron [2] has defined the concept of a c-compact closure space. In 1983, Mashhour and Ghanim stated an (equivalent) definition for this concept. [1, Theorem 4.1] and used the new definition to invistegate some properties of these spaces. Unfortunately, some of the results mentioned in [1] are incorrect. In this paper we give the correct version for compact closure spaces. Also we study some weak forms of compactness and give relations between these weak forms and regular closure spaces. Finally we give some relations between the weak forms of compactness and some type of continuous functions.

1.Preliminaries

In this section we collect, for sake of references, the needed definitions and results appeared in [1]. Also we give a corrected version for Theorem 4.1 in [1].

Definition 1-1: Let X be set. A function c: P(X)->P(X) is a Čech closure operator on X if it satisfies the following three axioms:-

C1: $c(\phi)=\phi$

C2: $c(A)\supset A$

C3: $c(A\cup B)=c(A)\cup c(B)$

The paire (X,c) is said to be a closure space.

A set $A \subset X$ is closed in (X,c), if $A=c(A)$. It is open if $X \setminus A$ is closed.

Definition 1.2 :- Let (X, c_1) be closure spaces. A function f:X⟶Y is said to be Č-continuous (resp. **Č-homeomorphism**) if $f(c_1(A)) \subset c_2(f(A))$ (resp. f is a bijection with $f(c_1(A))=c_2(f(A))$ for every $A \subset X$.

Remark 1.1:-

1) If c is a **Čech closure operator on a set** X, then the family $\tau(c)=\{X \setminus A : c(A)=A\}$ is a topology on X. But in general $c(A) \subset cl(A)$, where cl(A) is the $\tau(c)$-closure of A.
2) If f: $(X, c_1) \to (Y, c_2)$ is Č-continuous (resp. **Č-homeomorphism**), then $f:(X, \tau(c_1)) \to (Y, \tau(c_2))$ is continuous (resp. homeomorphism).

Definition 1.3 :- Let (X,c) be a closure space and let A be an arbitrary subset of X. The function $c_A : P(A) \to P(A)$, defined by :- $c_A(B)=A \cap c(B)$ is a Čech closure operator on A induced by c. The pair (A, c_A) is said to be a subspace of (X,c). It is closed (resp. open) subspace if $c(A)=A$ (resp. $c(X \setminus A)=X \setminus A$).)

Definition 1.4 :Let $\mathfrak{F}=\{(X_t, c_t) : t \in T\}$ be a family of pairwise disjaint closure spaces. If $X=\bigcup X_t$ then

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the Function $\amalg c_t : P(X) \to P(X)$ define by, $\amalg c_t(A) = \cup c_t\ (X_t \cap A)$, is a Čech closure operator on X. It is called the sum Čech closure operator on $X = \cup X_t$. The pair $(X, \amalg c_t)$ is called sum closure space of $\mathfrak{F}$.

Definition 1.5 :- Let $\mathfrak{F} = \{(X_t, c_t) : t \in T\}$ be a family of closure spaces. Let $X = \Pi X_t$ and let $\Pi c_t : P(X) \to P(X)$, defined as follows:-

For $A \subset X$ and $x \in X, x \in \Pi c_t(A)$ if the following condition is satisfied : $A = A_1 \cup \ldots \cup A_n (A_i \subset X) \Rightarrow$ there is i such that $x_t \in c_t(P_t(A_i))$ for all t, where $P_t : X \to X_t$ is the projection on the t-coordinate closure space . Then Πc_t is a Čech closure operator on X. It is called the **product Čech closure operator** on X. The pair $(X, \Pi c_t)$ is called the product closure space of $\mathfrak{F}$.

Remark 1.2 :- If the closure spaces are topological spaces under their closure operators, Definitions 1.2-1.5 are reduced to the corresponding definitions in ordinary topological spaces. In this case we say that Definitions 1.2-1.5 are **good** extensions.

Definition 1.6 :- Let (X,c) be a closure space. A family $\{A_j : j \in J\}$ of subsets of X is c-cover of X if $\{A_j^\circ : j \in J\}$ covers X, where $A^\circ = X \setminus c(X \setminus A)$. A closure space (X,c) is c-**compact** if every c-cover of X has a finite c-Subcover.

The following proposition is the corrected version of Theorem 4.1 in [1].

Proposition 1.1:-A closure space (X,c) is c-compact if and only if it satisfies the following condition :- Given any family $\mathfrak{F}$ of subsets of X such that the family $c(\mathfrak{F})=\{c(F):F\in\mathfrak{F}\}$ has the finite intersection property (FIP, for short), then $c(\mathfrak{F})$ has a non empty intersections.

proof :-To prove that the condition is necessary, Let (X,c) be a c-compact closure space and let $\mathfrak{F}=\{F_j:j\in J\}$ be a family of subsets of X such that the family $c(\mathfrak{F})=\{c(F_j):F_j\in\mathfrak{F}\}$ has FIP. Assume that $\cap\{c(F_j)\, j\in J\}=\phi$. Then $\{ X\setminus F_j:j\in J \}$ is a c-cover of X. Since X is c-compact, there is finite c-subcover $\{X\setminus F_j:j\in J^*\}$, i.e. $X=\cup\{(X\setminus F_j)^*:j\in J^*\}$. This implies that $\cap\{c(F_j):j\in J^*\}=\phi$; a contradiction. To prove that the condition is sufficient, let (X,c) be a closure space with the given condition and let $\{A_j:j\in J\}$ be a c-cover For X. The family $\{c(X\setminus A_j):j\in J\}$ has an empty intersection.Therefore, there exists a finite sub-family $\{c(X\setminus A_j):j\in J^*\}$ which has an empty intersection. This means that $\{A_j:j\in J^*\}$ is a finite c-subcover of $\{A_j:j\in J\}$

Remark 1.3 :- Every open cover for

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X (i.e. $X=\bigcup A_i, A_i=A_i^{\circ}$) is a c-cover and every c-cover for X is a cover for X.

Definition 1.7:- A closure space (X,c) is said to be **nearly c-compact** (resp. **almost c-compact**) if it satisfies the following condition :-

For every c-cover $\{A_i : i\in I\}$ of X, there is a finite set $J\subset I$ such that $\{c(A_i) : i\in J\}$ is a c-cover of X (resp. a cover for X).

Definition 1.8:- A closure space (X,c) is said to be **c-regular** if it satisfies the following condition:- $x\in A^{\circ} \Rightarrow$ there is $B\subset X$ such that $x\in B^{\circ}\subset c(B)\subset A^{\circ}$.

Definition 1.9:- A closure space (X,c) is said to be **regular** if it satisfies following condition :- $x\in A^{\circ} \Rightarrow$ there is $V\subset X$ such that $x\in V^{\circ}\subset c(V)\subset A$.

Proposition 1.2:-In c-regular closure spaces, c-compactness $\Rightarrow$ almost c-compactness.

Definition 1.10 :- A function f: X$\longrightarrow$ Y, where X and Y are closure spaces, is said to be $\check{C}$ θ-continuous if for each point $x\in X$ and for each set $V\subset Y$ such that $f(x)\in V^{\circ}$, there is $U\subset X$ such that $x\in U^{\circ}$ and $f(c(U))\subset c(V)$.

Proposition 1.3:- The $\check{C}$ θ-continuous image of an almost c-compact closure space is almost c-compact.

Definition 1.11:- A function f: X $\longrightarrow$Y, where

X and Y are closure spaces, is said to be $\check{C}$ - **almost continuous** (or $\check{C}$ A-continuous, for short) if for each $x \in X$ and each $V \subset Y$ with $f(x) \in V^{*}$, there is $U \subset X$ such that $x \in U^{*}$ and $f(U) \subset (c(V))^{*}$.

Proposition 1.4:- A $\check{C}$ A-continuous image of a c-compact closure space is nearly c-compact.

2. Compact and compact* closure spaces.

In this section we introduce new definitions for compact closure spaces. Using these new definitions (which are good extensions) we give corrected versions for Theorems 4.2-4.5 in [1]

Definition 2.1:- A closure space (X,c) is said to be compact if every c-cover for X has a finite subcover.

Definition 2.2:- A closure space (X,c) is said to be compact^{*} if every open cover for X has a finite subcover.

Remark 2.1:- Let (X,c) be a closure space. Let "cl" and "int" be the closure and the interior operators respectively in topology $\tau(c)$ induced by the Čech closure operator c. It is clear that a set A is closed if and only if A= c(A)=cl(A). Also a set A is open if and only if $A^{*}=int(A)=A$.

Proposition 2.1:-A closure space (X,c) is compact if and only if it satisfies the following condition:- Given any family $\mathfrak{S}$ of subsets of X with

FIP, then the family c($\mathfrak{F}$) has a non empty intersection.

Proof. It is similar to that of Proposition 1.1.

Proposition 2.2:- A closure space (X,c) is compact^{+} if and only if every family of closed sets with FIP has non empty intersection.

Proof :- The result follows from the fact that (X,c) is a compact^{+} closure space if and only if $(X,\tau(c))$ is a compact topological space. and Remark 2.1.

Remark 2.2:-

1) c-compactness, compactness and compactnessΩ^* are good extensions.

2) c-compactness → compactness → compactnessΩ^*,

Using similar arguments as in [1] one may prove the following:-

Proposition 2.3:- A $\check{C}$ -continuous image of a compact (resp. compact^{+}) closure space is compact (resp. compact^{+}).

Proposition 2.4 :- A closed subspace of a compact (resp. compact^{+}) closure space is compact (resp. compact^{+}).

Proposition 2.5 :- A finite sum closure space is compact(resp. compact^{+})if and only if each coordinate closure space is compact(resp.compact^{+})

Proposition 2.6 :- A product closure space is compact (resp. compact^{+})if and only if each coordinate closure space is compact (resp.compact^{+})

Remark 2.3:- Propositions 2.2 -2.5 give corrected versions for Theorems 4.2-4.5 in [1].

3. Weak forms of compactness in closure spaces.

In this section we use Definitions 2.1 and 2.2 to introduce some weak forms of compactness in closure space which are all good extensions. The relations between these different forms of compactness are indicated. We include this section by introducing good extensions for regular topological spaces to closure spaces. The relations between these new concepts and compactness are investigated.

Definition 3.1 :- A closure space (X,c) is said to be :-

1) nearly compact if for every c-cover $\{A_i : i\in I\}$ of X there is a finite set $J\subset I$ such that $\{(c(c(A_i)))^{\circ} : i\in J\}$ covers X.

2) S-nearly compact' if for every open cover $\{A_i : i\in I\}$ of X there is a finite set $J\subset I$ such that $\{(c(A_i))^{\circ} : i\in J\}$ covers X.

3) nearly compact' if for every open cover $\{A_i : i\in I\}$ of X there is a finite set $J\subset I$ such that $\{(c(c(A_i)))^{\circ} : i\in J\}$ covers X.

4) almost compact if for every c-cover over $\{A_i : i\in I\}$ of X there is a finite set $J\subset I$ such that $\{c(c(A_i)) : i\in J\}$ covers X.

5) S-almost compact* if for every open cover $\{A_i : i\in I\}$ of X there is a finite set $J\subset I$ such that

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$\{c(A_i): i \in J\}$ covers X

6) almost compact' if for every open cover $\{A_i : i \in I\}$ of X there is a finite set $J \subset I$ such that $\{c(c(A_i)): i \in J\}$ covers X

Proposition 3.1:- The following implications hold.

1) c-compact → nearly c-compact → almost c-compact [1].
2) compact (resp. nearly compact) → almost compact.
3) compact' → nearly c-compact' → almost compact'.
4) compact' → S-nearly c-compact' → S-almost compact'
5) S-nearly compact' (resp. S-almost compact') → nearly compact' (resp. almost compact').
6) c-compact (resp. nearly c-compact, almost c-compact) ⇒ compact (resp. nearly compact, almost compact) ⇒ compact' (resp. nearly compact' ,almost compact').
7) c-compact (resp. nearly c-compact, almost c-compact) ⇒ compact' (resp. s-nearly compact' , s-almost compact').
8) compact ⇒ almost c-compact.

Proof : Straightforward.

Remark 3.1: One may notice that in general, compact → nearly compact. This is a remarkable deviation from the case of topological spaces.

In addition to Definitions 1.8-1.9, the following definition gives good extensions for the concept of a regular topological space.

Definition 3.2: A closure space (X,c) is said to be:

i) regular$'$ if $x \in A$, A open $\Rightarrow$ there is $B \subset X$, B open such that $x \in B \subset c(B) \subset A$.

ii) S_1 - regular if $x \in A^{\circ} \Rightarrow$ there is $B \subset X$ such that $x \in B^{\circ} \subset c(B) \subset c(c(B)) \subset A$.

iii) S_2- regular if $x \in A^{\circ} \Rightarrow$ there is $B \subset X$ such that

$x \in B^{\circ} \subset c(B) \subset c(c(B)) \subset A^{\circ}$.

iv) S-regular$'$ if $x \in A$, A open $\Rightarrow$ there is $B \subset X$, B open such that $x \in B \subset c(B) \subset c(c(B)) \subset A$.

One may verify the following implications :

$$\begin{array}{ccccc} \text{S-regular}' & \Leftarrow & S_2\text{-regular} & \Rightarrow & S_1\text{-regular} \\ \Downarrow & & \Downarrow & & \Downarrow \\ \text{regular}' & \Leftarrow & \text{c-regular} & \Rightarrow & \text{regular} \end{array}$$

Using similar arguments as in [1] one may prove the following:

Proposition 3.2 : In regular$'$ space, (X,c) is compact$'$ $\Leftrightarrow$ (X,c) is S-almost compact$'$.

Proposition 3.3:- In S_1-regular closure space, **compactness $\Leftrightarrow$ almost compactness.**

Proposition 3.4: In S_2 -regular closure space, c-compactness $\Leftrightarrow$ almost compactness.

Proposition 3.5: In S-regular$'$ closure space compactness$'$ $\Longleftrightarrow$ almost compactness.

In addition to Definition 1.10, the following definition gives good extensions for the concept of θ-continuous functions between topological spaces.

Definition 3.3: A function $f: (X,c_1) \rightarrow (Y,c_2)$

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is:

i) S $\check{C}$ θ- continuous if for each point $x \in X$ and for each set $V \subset Y$ such that $f(x) \in V^{*}$, there is $U \subset X$ such that $x \in U^{*}$ and $f(c_1(c_1(U)) \subset c_2(c_2(V))$

ii) $\check{C}$ θ- continuous' if for each point $x \in X$ and for each open set $V \subset Y$ such that $f(x) \in V$, there is open set $U \subset X$ such that $x \in U$ and $f(c_1(U)) \subset c_2(V)$.

iii) S $\check{C}$ θ- continuous' if for each point $x \in X$ and for each open set $V \subset Y$ such that $f(x) \in V$, there is open set $U \subset X$ such that $x \in U$ and $f(c_1(c_1(U)) \subset c_2(c_2(V))$.

Using similar argument as in [1] one may prove the following:

Proposition 3.6: The S $\check{C}$ θ- continuous image of an almost compact closure space is almost compact.

Proposition 3.7: The $\check{C}$ θ- continuous' image of an S-almost compact' closure space is S-almost compact'.

Proposition 3.8: The S $\check{C}$ θ- continuous image of an almost compact' closure space is almost compact'.

In addition to Definition 1.11, the following definition gives good extensions for the concept of almost continuous functions between topological spaces.

Definition 3.4: A function f: $(X, c_1) \longrightarrow (Y, c_2)$

is:

i) S $\check{C}$ A- continuous if for every $x \in X$ and each $V \subset Y$ with $f(x) \in V^{*}$ there is $U \subset X$ such that $x \in U^{*}$ and $f(U) \subset (c_2(c_2(V)))^{*}$.

ii) $\check{C}$ A- continuous' if for every $x \in X$ and each open set $V \subset Y$ with $f(x) \in V$ there is an open set $U \subset X$ such that $x \in U$ and $f(U) \subset (c_2(V))^{*}$.

iii) S $\check{C}$ A- continuous' if for every $x \in X$ and each open set $V \subset Y$ with $f(x) \in V$ there is an open set $U \subset X$ such that $x \in U$ and $f(U) \subset (c_2(c_2(V)))^{*}$.

Using similar arguments as in [1] one may prove the following:

Proposition 3.9: The $\check{C}$ A- continuous image of a compact closure space is nearly c-compact.

Proposition 3.10: A S $\check{C}$ A- continuous image of compact (c-compact) closure space is nearly compact.

Proposition 3.11:- The $\check{C}$ A-continuous' image of a compact' closure space is nearly compact*.

Proposition 3.12:- The S $\check{C}$ A-continuous image of a compact' closure space is nearly compact'

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عن فراغات الاحكام المتراصة

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فى عام ١٩٨١ قدم ثرون تعريفا لمفهوم فراغ الاحكام المتراص -C · وفى عام ١٩٨٣ قدم مشهور وغانم تعريفا (مكافئا) لهذا المفهوم ، واستخدما هذا التعريف الجديد فى دراسة بعض خواص فراغات الاحكام المتراص -C · وبدراسة النتائج التى قدمها مشهور وغانم فى بحثهما المشار اليه ، وجدنا أن بعض تلك النتائج غير صحيحة·

فى هذا البحث نقدم - من وجهة نظرنا - رؤية صحيحة لفراغات الاحكام المتراصة· أيضا نقدم دراسة لبعض الصور الضعيفة من خاصيه التراص ، وعلاقات هذه الصور الضعيفة من خاصية التراص بفراغات الاحكام المنتظمة ، وأخيرا أوضحنا بعض العلاقات الموجودة بين الصور الضعيفة لخاصية التراص وبين أنواع من الدوال (المتصلة)·