Delta J.Sci.16 (2) 1992, 1 - 10

ON THE GENERALIZED LINEAR SEARCH PROBLEM

BY

*A.B. El-Rayes ,**M. A. Abd El-Moneim & H.A. Fergani**

* Military Technical College, Cairo, Egypt.

** Mathematics Department, Faculty of Science, Tanta University

Tanta - Egypt

Received: 4-4-1991

ABSTRACT

A target is located on a straight line, its position is given by the random variable X_o, which has distribution function F, we start searching from some point on the line following a path $\phi(t)$. In this paper conditions under which we can minimize the expected time of finding the target will be given and we find the optimal path which minimizes expected value in some special cases of F.

I- INTRODUCTION

Let X_o be the position of a randomly located target on the real line. It is assumed that the expectation of X_o is finite. A searcher starts from any point a_o, $|a_o| < \infty$), on the real line to detect the target. He moves continuously along the line in both directions of the searching point a_o. If a_o is equal to zero we obtain the so called linear search problem. But if a_o is not equal to zero the problem is called generalized linear search problem.

Let $\{a_i\}_{i \in Z}$ be a sequence of real numbers with $a_{2i+1} \longrightarrow d$ and $a_{2i} \longrightarrow c$ where c and d are defined as follows:

$c = \inf \{t : F(t) > 0\}$, $d = \sup \{t, F(t) < 1\}$, and Z is the set of integers

Delta J. Sci. 16 (2) 1992

دراســة مشكلة البحــث الخطية فى الحالـة العامـة

++احمد بهاء الدين الريس +عبد المنعم انور +هالة فرجانـى
+ قسم الرياضيات ـ الكلية الفنية العسكـرية
++ قسم الرياضيات ـ كلية العلوم ـ جامعة طنطـا

موضع جسيم على خط مستقيم يعرف بالمتغير العشوائى X_0 والذى له دالة التوزيع F . بدأنا البحث عن هذا الجسيم من اى نقطة على الخط المستقيم تابعين مسـارا $\phi(t)$. فى هذا المقال تم ايجاد الشروط التى تجعل القيمـة المتوقعة لزمن اكتشـاف الجسـيم اقل ما يمكـن (حيث أن هـذا الزمن متغــير عشـوائى) . وكذلك تم ايجاد المسـار المثالى الذى يجـعل تلك القيمـة المتوقعـة اقل مـا يمكـن وذلك لبعـض الحالات الخاصـة للدالة F .

We consider the following cases according to the position of the searcher at the begining of the search.

Case (0) In this case we have

$$\cdots \leqq a_4 \leqq a_2 \leqq a_o = o \leqq a_1 \leqq a_3 \leqq \cdots$$

Case (1) This case consists of all search paths such that

$$\cdots \leqq a_4 \leqq a_2 \leqq a_o \leqq o \leqq a_1 \leqq a_3 \cdots$$

Case (2) In this case we have

$$\cdots \leqq a_4 \leqq a_2 \leqq o \leqq a_o \leqq a_1 \leqq a_3 \leqq \cdots$$

Case (3) Let I be a finite and nonempty set of odd numbers. for $j \in I$ we have:

$$\cdots \leqq a_{j+1} \leqq a_{j-1} \leqq \cdots \leqq a_2 \leqq a_o \leqq a_1 \leqq \cdots \leqq$$
$$\leqq a_{j-2} \leqq a_j \leqq o \leqq a_{j+2} \leqq a_{j+4} \leqq \cdots$$

Case (4) In this case we consider I to be a finite and nonempty set of even numbers. For $j \in I$ we have

$$\cdots \leqq a_{j+4} < a_{j+2} \leqq o \leqq a_j \leqq a_{j-2} \leqq \cdots \leqq a_2 \leqq a_o \leqq a_1 \leqq a_3 \leqq \cdots$$

Denote by Φ_k, k=o,1,2,3,4, the class of search plans in case k. If k=o we obtain the linear search problem.

The total distance travelled from the starting point to the target position is a random variable, if the searcher's speed is equal to 1 then this total distance is equal to the time of finding the target. Let $\tau_k(\psi)$ be the time of finding the target in the case k, where ψ is a search plan which is defind by a sequence $\{a_i\}_{i \in Z}$ and $\psi \in \Phi_k$. Denote by $E[\tau_k(\psi)]$ the expected value of the time of finding the target. We aim to

Delta J. Sci. 16 (2) 1992

Rayes et al.

find $\psi^* \in \Phi_k$ which makes $E[\tau_k(\psi)]$ minimum, that is

(1) $E[\tau_k(\psi^*)] \leqq E[\tau_k(\psi)], \quad \forall \psi \in \Phi_k$

we call ψ^* optimal search path.

Necessary conditions under which the search path is optimal and optimal paths for certain distributions have been found in the linear search problem see $\{$[1], [3], [4], [5] and [6]$\}$.

In the generalized linear search problem some conditions unde which the expected time of finding the target is minimum would be found for certain distributions see ([2]). Here we find conditions for optimality for any distribution and the optimal paths will be find in some special cases of F.

II- Search models and conditions of optimality

Let $\psi = \{a_i\}_{i \in Z}$ be a search plan, where $\psi \in \Phi_k$ k=o,1,2,3,4 the expected value of the time of finding the target can be calculated, it is given by: (see [1] and (2])

(2) $E[\tau_k(\psi)] = E|X| + D_k(\psi)$ where

(3) $$D_o(\psi) = 2\sum_{i=1}^{\infty}\{a_{2i-1}(1-\nu[a_{2i-2}, a_{2i-1}]) + |a_{2i}|[1-\nu[a_{2i}, a_{2i-1}])\}$$

(4) $$D_1(\psi) = A + |a_o| + 2\sum_{i=1}^{\infty}\{a_{2i-1}(1-\nu[a_{2i-2}, a_{2i-1}]) + |a_{2i}|(1-\nu[a_{2i}, a_{2i-1}])\}$$

(5) $$D_2(\psi) = A - |a_o| + 2\sum_{i=1}^{\infty}\{a_{2i-1}(1-\nu[a_{2i-2}, a_{2i-1}]) + |a_{2i}|.(1-\nu[a_{2i}, a_{2i-1}])\}$$

(6) $$D_3(\Psi) = A + |a_o| - 2|a_j|\{1 - \nu[a_o, a_j]\} + \sum_{i=1}^{(j-1)/2} (|a_{2i}| - |a_{2i-1}|)\{1 - \nu[a_o, a_j]\} + \sum_{i=j+1}^{\infty} \{a_{2i-1}(1 - \nu[a_{2i-2}, a_{2i-1}]) + |a_{2i}|(1 - \nu[a_{2i}, a_{2i-1}])\}$$

(7) $$D_4(\Psi) = A - |a_o| - 2|a_j|\nu(a_j, a_o) + 2\sum_{i=1}^{j/2} (|a_{2i-1}| - |a_{2i}|)\{1 - \nu(a_1, a_o)\} + 2\sum_{i=j+1}^{\infty} \{a_{2i-1}(1 - \nu[a_{2i-2}, a_{2i-1}]) + |a_{2i}|(1 - \nu[a_{2i}, a_{2i-1}])\}$$

where:

$$A = \begin{cases} -2\int_o^{a_o} |x|\, dF(x), & \text{if } X \text{ is continuous random variable,} \\ -2\sum |x|\, f(x), & \text{if } X \text{ is discrete random variable, } f(x) \text{ is the probability distribution of } X \end{cases}$$

and ν is a measure of probability induced by the random variable X, $\nu(x,y) = F(y) - F(x)$, $\nu(c,d) = 1$.

For any $y, z \in R$ we define the function $g(y,z)$ as follows:

$$g(y,z) = \frac{\nu(y,z)}{z - y},$$ where R is the set of real numbers.

Theorem 2.1: Let Ψ be a search plan, $\Psi = \{a_i\}_{i\in Z}$, $\Psi \in \Phi_k$, $k=o,1,2,3,4$ then the necessary condition which makes Ψ optimal is:

(8) $g(a_{2i-1}, a_{2i+1}) > g(a_{2i+2}, a_{2i}) > g(a_{2i+1}, a_{2i+3})$

Proof: We give the proof only in one case, say k=2, the proof for the other cases can easily be given in the same way. Let $\Psi \in \Phi_2$, the expected time of finding the target in this case be given by equation (5)

$$D_2(\Psi) = A - |a_o| + 2 \sum_{i=1}^{\infty} \{a_{2i-1}(1-\nu[a_{2i-2}, a_{2i-1}]) + $$

$$+ |a_{2i}| (1-\nu[a_{2i}, a_{2i-1}])\}.$$

we define other search plan $\psi' \in \Phi_2$ which is defined by the sequence $\{a_i'\}_{i \in Z}$, $i \geqq o$ such that:

$$(9) \quad a_i' = \begin{cases} a_i & i \leqq 2\ell - 1 \\ a_{i+2} & i \geqslant 2\ell \end{cases}$$

we assume that Ψ is optimal search plan and condition $g(a_{2\ell-1}, a_{2\ell+1}) \leqq g(a_{2\ell+2}, a_{2\ell})$ is satisfied. By substracting $D_2(\Psi) - D_2(\psi')$ we get

$$D_2(\Psi) - D_2(\Psi') = 2\{|a_{2\ell}|(1-\nu[a_{2\ell}, a_{2\ell-1}]) + |a_{2\ell+2}| (1-\nu[a_{2\ell+2}, a_{2\ell+1}]) +$$

$$+ a_{2\ell+1}(1-\nu[a_{2\ell}, a_{2\ell+1}]) - |a_{2\ell+2}| (1-\nu[a_{2\ell+2}, a_{2\ell-1}]\}$$

$$= 2\{|a_{2\ell}|(1-\nu[a_{2\ell}, a_{2\ell-1}] - \nu[a_{2\ell-1}, a_{2\ell+1}] + \nu[a_{2\ell-1}, a_{2\ell+1}]) +$$

$$+ |a_{2\ell+2}| (\nu[a_{2\ell+2}, a_{2\ell-1}] - \nu[a_{2\ell+2}, a_{2\ell-1}] - \nu[a_{2\ell-1}, a_{2\ell+1}]) +$$

$$+ a_{2\ell+1}(1-\nu[a_{2\ell}, a_{2\ell+1}] - \nu[a_{2\ell+2}, a_{2\ell}] + \nu[a_{2\ell+2}, a_{2\ell}])\}$$

$$= 2\{|a_{2\ell}|(1-\nu[a_{2\ell},a_{2\ell+1}] + \nu[a_{2\ell-1},a_{2\ell+1}])$$

$$-|a_{2\ell+2}|\nu[a_{2\ell-1},a_{2\ell+1}] + a_{2\ell+1}(1-\nu[a_{2\ell+2},a_{2\ell+1}]$$

$$+\nu[a_{2\ell+2},a_{2\ell}])$$

$$= 2\{|a_{2\ell}|(1-\nu[a_{2\ell},a_{2\ell+1}]) + \nu(a_{2\ell-1},a_{2\ell+1})(|a_{2\ell}| - |a_{2\ell+2}|) +$$

$$+ a_{2\ell+1}(1-\nu[a_{2\ell+2},a_{2\ell+1}]) + a_{2\ell+1}\nu[a_{2\ell+2},a_{2\ell}]$$

$$- a_{2\ell-1}\nu[a_{2\ell+2},a_{2\ell}] + a_{2\ell-1}\nu[a_{2\ell+2},a_{2\ell}]$$

$$> 2\{\nu(a_{2\ell+2},a_{2\ell})[a_{2\ell+1} - a_{2\ell-1}] - \nu(a_{2\ell-1},a_{2\ell+1})[a_{2\ell} - a_{2\ell+2}]\}$$

$$= 2[a_{2\ell+1} - a_{2\ell-1}][a_{2\ell} - a_{2\ell+2}]\left[\frac{\nu(a_{2\ell+2},a_{2\ell})}{a_{2\ell} - a_{2\ell+2}} - \frac{\nu(a_{2\ell-1},a_{2\ell+1})}{a_{2\ell+1} - a_{2\ell-1}}\right]$$

$$= 2[a_{2\ell+1} - a_{2\ell-1}][a_{2\ell} - a_{2\ell+2}][g(a_{2\ell+2},a_{2\ell}) - g(a_{2\ell-1},a_{2\ell+1})]$$

$$\geqq 0$$

i.e. $D_2(\psi) \geqq D_2(\psi')$ which contradicts that ψ is optimal search plan.

The proof is complete.

III - Optimal search paths for some distributions

Theorem 3.1 : Let $f(x)$ be the density function of X_o. If $f(x)$ is monotonic increasing in the interval $[a_o,d]$ and monotonic decreasing in the interval $[c,a_o]$, then the optimal search plan of any case consists of two end points.

Proof: We prove this theorem when k=2 (or in the case (2)). Let a search plan consists of $\{c,a_2,a_o,a_1,d\}$. From necessary condition of optimality we obtain

$$\frac{\nu(a_o,a_1)}{a_1 - a_o} > \frac{\nu(a_1,d)}{d - a_1} \quad , \text{ then}$$

$$(10)\qquad \frac{F(a_1) - F(a_0)}{a_1 - a_0} > \frac{F(d) - F(a_1)}{d - a_1}$$

From (10) we get:

$$(11)\qquad f(\theta_1) > f(\theta_2) \quad , \quad a_0 < \theta_1 < a_1, \; a_1 < \theta_2 < d,$$

where f is the density function of the position of the target, this contradicts the assumption of our theorem.

We can conclude that a_1 must be equal to d also the optimal search plan consists of the two end points, the other cases can be proved by the same manner.

Lemma: If $f(x)$ is a function decreasing (or increasing) on the interval $[c,d]$, $-\infty \leqq c < 0 < d \leqq +\infty$, $\int_c^d f(x)dx = 1$ then:

$$\int_{a_0}^{y} f(x)dx \leqq \frac{y - a_0}{y + |c|} \quad \text{if } y \geqq a_0 \quad \text{if } f(x) \text{ is decreasing.}$$

$$\int_{y}^{a_0} f(x)dx \leqq \frac{|y| - a_0}{|y| + d} \quad \text{if } y \leqq a_0 \quad \text{if } f(x) \text{ is increasing}$$

Proof: Let $f(x)$ is a decreasing function on $[c,d]$

$$\int_{a_0}^{y} f(x)dx = 1 - \int_{c}^{a_0} f(x)dx - \int_{y}^{d} f(x)dx$$

$$\leqq 1 - \int_{c}^{a_0} f(x)dx \leqq 1 - f(a_0)(a_0 - c).$$

But $\int_{a_0}^{y} f(x)dx \leqq f(a_0)(y - a_0)$ if $y > a_0$ ($y = a_0$ is trivial)

$$f(a_0) \geqq \frac{1}{y - a_0} \int_{a_0}^{y} f(x)\,dx \text{ then } \int_{a_0}^{y} f(x)dx \leqq 1 - \frac{a_0 - c}{y - a_0} \int_{ao}^{y} f(x)dx$$

$$\int_{a_o}^{y} f(x)dx \leqq \frac{y - a_o}{y + |c|}$$

(by similar manner we can prove the other cases).

Theorem 3.2 : Suppose that the position of the target has a density function f(x) which is monotonic on the interval [c,d], then the optimal search plan, in all cases, (k = o,1,2,3,4), is $\{c,d\}$ if f(x) is decreasing function,and $\{d,c\}$ if f(x) is increasing function:

Proof: Consider f(x) as a decreasing function (the case of increasing function is similar), We shall give the proof only in one case, say k = 2. The optimal search plan consists of at most 3 points. In order to prove this take for example a search plan $\psi_1 = \{a_1, a_2, d, c\}$ we can obtain $g(a_1, d) \leqq g(c, a_2)$ which contradicts the condition (8) for i = 1. Suppose a search plan ψ which is defined by $\{y, c, d\}$, $y \geqslant a_o$ and other search plan $\psi' = \{c,d\}$ we obtain;

$$\begin{aligned}
E[\tau_2(\psi)] &= E|X| + A - |a_o| + 2[y(1-\nu(a_o,y)) + |c|\{1-\nu(c,y)\}] \\
&= E|X| + A - |a_o| + 2[y - y\nu(a_o,y) + |c|\nu(a_o,d) - \\
&\quad - |c|\nu(a_o,y)] \\
&= E|X| + A - |a_o| + 2[y - \nu(a_o,y)(y + |c|) + \\
&\quad + |c|\nu(a_o,d)] \\
&\geqq E|Y| + A - |a_o| + 2[y - \frac{(y-a_o)}{y+|c|}(|y| + |c|) + \\
&\quad + |c|\nu(a_o,d)]
\end{aligned}$$

$$= E|X| + A - |a_o| + 2[\ a_o + |c|\nu(a_o,d)]$$

$$\geqq E|X| + A - |a_o| + 2[\,|c|\nu(a_o,d)]$$

$$= E[\tau_2(\psi)]$$

the proof is complete.

REFERENCES

1- A.B. El-Rayes and M.A. Abd El-Moneim, "Searching of Randomly Located Target, the first ORMA conference, Egypt 1(1984) 225-229

2- Balki, Z. " The Generalized linear search problem,.Existence of optimal search paths", J. of the oper. research, Japan, Vol. 30, No. 4, (1987) 399-421.

3- Beck, A. " On the linear search problem ",Israel J. Math. 3 (1964), 221 - 228.

4- Beck, A. "More on the linear search problem " Israel J. Math. 3 (1965), 61-70.

5- Beck, A. and Newman, D. " Yet more on the linear search problem", Israel, J. Math. 8(1970), 419-429.

6- Stone, L. " Theory of optimal search " , Academic press, New York, 1975.