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A GENERALIZED LIAPUNOV FUNCTION WITH UNIFORM STABILITY OF SOLUTIONS OF DIFFERENTIAL EQUATIONS

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ABSTRACT

A theory of asymptotic stability via Liapunov functions developed for the non-autonomous differential equations

$$X' = f\ (t, X). \qquad (1)$$

The concept of uniformly Lipschitz stability is discussed and a new sufficient condition is given for both uniformly asymptotic and lipschitz stability of the zero solution of (1).

1. INTRODUCTION

In 1960 's Yoshizawa [10] and La salle [5] developed a general theory for examining the asymptotic stability of the zero solution of the non-autonomous system

$$X' = f\ (t,X) \tag{1}$$

Where $f : [0, \infty) \times R^n \rightarrow R^n$ is continuous. The function f is smooth enough to ensure existence and unigueness of solutions of (1) through every point (t_o , x_o) and $f(t,o) = o$.

The main tool used was to emply properties of the derivative of a Liapunov function in order to locate limit sets of solutions of the system (1). Recently the boundedness condition of La salle [5] on the function f(t,x) was relaxed by Burton [1] which was followed by several improvements like as Haddock [3] and Peterson [9]. Meanwhile the uniformly asymptotic stability conditions for autonomous system of Barbashin and Krsovskii were recently generalized to the nonautonomous equation (1) by Hatvani [4], Matrosov [6], Wada and Yamamoto [8], ..., etc.. In these recent papers Dannan and El-Ayadi [2] introduced a uniform Lipschitz stability theory. They showed by an interesting example that uniform Lipschitz stability is not implied by uniform asymptotic stability in general.

The aim of this paper is to give sufficient conditions, for global attractivity of the zero solution of (1) by imporving the conditions on the derivative of Liapunov function and to give sufficient conditions under which uniform Lipschitz stability implied by uniform asymptotic stability.

In section 2, we improve the result work of the Haddock [3] and give an example which shows that our result is a generalization of the result of Haddock [3].

In section 3 we discuss the sufficient conditions under which the zero solution of (1) satisfies both uniform asymptotic and Lipschitz stability in the same time. This is unlike the conjecture of [2].

2- A generalized liapunov function

In this section we discuss the sufficient conditions which guarantee the asymptotic stability of the zero solution of (1). Following [3], we shall say that a continuous scaler function V defined on $[0, \infty) \times R^n$ is a liapunov function for (1) if :

(i) V has continuous first partial derivatives,

(ii) for any compact set H in R^n, V (t,X) is bounded from below for all $t \geq 0$ and $X \in H$, and

(iii) $V(t,X) = \text{grad } V.\ f(t,X) + \frac{\partial V}{\partial t}$

for $(t,X) \in [0,\infty) \times R^n$

To a closed set H_∞ in R^n, $H = H \cup \{\infty\}$ for $t \geq 0$. Let $X \in k_1 \subset R^n$, k_1 is compact subset of R^n, $S^c(H, \varepsilon)$ is the complement of ε-neighbourhood of H.

<u>THEOREM 2.1</u> Let X(t) be a solution of (1) with maximal right interval of definition $[t_o, T)$. Suppose that there exists a closed set H_∞ in R^n, a Liapunov function V(t,X) satisfying

$$V'(t,X) \leq - |f(t,X)| + V(t,X), \qquad (2.1)$$

for $X \in H \cap S^c(H, \varepsilon)$, $t \in [t_o, T)$,

and

$$\int_{t_n^*}^{t_n^{**}} V(s,X(s))ds < \infty \qquad (2.2)$$

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for any subinterval $[t_n^*, t_n^{**}] \subset [t_o, T)$.

If each solution of (1) approaches a constant in H, then $X(t) \to H_\infty$ as $t \to T-$, where $T \leq \infty$ is the right-hand endpoint of the interval of definition of X(t).

Proof. Let X(t) be a solution of (1) with maximal right-interval of definition $[t_o, T)$, Γ^+ denotes the set of positive limit points of X(t), thus we have two cases

(1) If Γ^+ is non-empty set, then there exist a regular points $z_i \in \Gamma^+$ for each integer number i and the solution $X(t) \to z_i$ as $t \to T-$ for all $z_i \in \Gamma^+$, i.e.

$$X(t) \to \Gamma^+ \quad \text{as } t \to \infty. \tag{2.3}$$

(2) If Γ^+ is empty set, then there is no point in Γ^+, the solution X(t) tends to it, and therefore X(t) has no limit in Γ^+, i.e.

$$X(t) \to \infty \qquad \text{as } t \to T-\ , \tag{2.4}$$

since by definition

$$\Gamma^+_\infty = \Gamma^+ \cup \{\infty\},$$

from the above discussion of (2.3) and (2.4) we have.

$$X(t) \rightarrow \Gamma^+_\infty$$

To show that $X(t) \rightarrow H_\infty$ as $t \rightarrow \infty$, it suffices to prove that $\Gamma^+ \subseteq H_\infty$.

Let $P \in \Gamma^+$, $P \notin H$, then $X(t) \not\rightarrow P$ as $t \rightarrow \infty$, otherwise, this contradicts the assumption that each solution of (1) approaches to a constant in H.

Now, let $q \in \Gamma^+$, $q \neq p$ be another point in Γ^+. Setting $\varepsilon = \frac{1}{2}\min\,(d(p,q),\, d(p,H)) > 0$,
where $d(p,q) = |p-q|$,

$$d(p,H) = \inf\{d(p,y) : y \in H\}.$$

Since $\varepsilon = \frac{1}{2}\min\,(d(q,p),\, d(p,H)) > 0,\ p,q \in \Gamma^+$,

$$\overline{S(p,\varepsilon)} \subseteq S^c(H,\varepsilon), \quad P,q \in \Gamma^+$$

there exist two sequences $\{t_n^*\} \rightarrow \infty$ and $\{t_n^{**}\} \rightarrow \infty$,

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i.e., $X(t_n^*) \to p$ as $n \to \infty$ and $X(t_n^{**}) \to q$ as $n \to \infty$

such that

$$|X(t_n^*) - X(t_n^{**})| \geq \varepsilon/2 \quad \text{for } t_n^* \leq t \leq t_n^{**}, \tag{2.5}$$

$$\text{for } X(t) \in \overline{S(P, \varepsilon)} \cap S^c(H, \varepsilon) = \overline{S(P, \varepsilon)},\ t_n^* \leq t \leq t_n^{**}, \tag{2.6}$$

Now, by the assumption (2.1) that any solution $X(t)$ of (1) satisfies (2.6), we have

$$V'(t,X(t)) \leq -|f(t,X(t))| + V(t,X(t)) , \tag{2.7}$$

By using the known technique [3] by dividing the interval $[t_o,T)$ into k equal subintervals $[t_n^*, t_n^{**}]$, for any positive integer k, an integration of (2.7), gives

$$V(t_k, X(t_k)) - V(t_o, X(t_o)) \leq \int_{t_o}^{t_k} V'(s,X(s))\, ds$$

$$\leq -\int_{t_o}^{t_k} |f(s,X(s))|\, ds + \int_{t_o}^{t_k} V(s,X(s))\, ds$$

i.e.

$$V(t_k, X(t_k)) - V(t_o, X(t_o)) \le - \sum_{n=1}^{k} \int_{t_n^*}^{t_n^{**}} | f(s,X(s)) | ds + \sum_{n=1}^{k} \int_{t_n^*}^{t_n^{**}} V(s,X(s))ds$$

Thus, by (1) , (2.2) and (3.5) we obtain

$$V(t_k, X(t_k)) - V(t_o, X(t_o)) \le -\infty \quad .$$

This contradicts the assumption that V(t,X) is bounded below. Hence, $\Gamma^+ \subseteq H_\infty$, this completes the proof.

The following example is an application of Theorem (2.1) which it is not covered by Haddock [3].

Example : Consider the second order system.

$$x' = q(t)\, y + P(t)\, x$$

$$y' = - q(t)\, y - p(t)\, x$$

Where, for $t \ge o$, P and q are continuous, $P(t) \ge o$ and $|q(t)| \le \beta P(t)$ for some $\beta > o$. Define $V = 1/2 (x^2 + y^2)$. Then $V' = -2 P(t) (y^2 + V)$. Let H be the set of points on

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the x - axis, for each $\theta > o$ and each compact set G in R^n, we have $V' \leq -P(t)\varepsilon^2 + V$ and $|f(t,x)| \leq \varepsilon P(|x| + |y|) \leq P(t)$ for all $t \geq 0$, all $(x,y) \in K \cap S^c(H,\varepsilon)$ and some $\varepsilon = \beta + 1 > 0$ which depend on t. Thus it is clear that $V'(t,x,y) \leq -\frac{\theta^2}{\varepsilon}|f(t,x,y)| + V$ for $t \geq o$ and $(x,y) \in K \cap S^c(H,\varepsilon)$ thus the condition (2.1) is satisfied. The remaining condition is already satisfied as in Haddock [3].

3- Uniformly asymptotic-Lipschitz stablility

In this section we discuss the concepts of uniformly asymptotic and uniformly Lipschitz stability.

Definition 1. : The solution x(t) of (1) is said to be uniformly asymptotically stable, if for each $\varepsilon > o$, there exists a $\delta = \delta(\varepsilon) > o$ such that, for any solution $\bar{x}(t) = x(t, t_o, x_o)$ of (1), the inequalities:

$$\|\bar{x}(t_1) - x(t_1)\| \leq \varepsilon \text{ and } t_1 \geq t_o,$$

imply

$$\|\bar{x}(t) - x(t)\| < \varepsilon \text{ for all } t \geq t_1 + T$$

Where $T = T(\varepsilon)$

Definition 2. A function $Q(r)$ is said to belong to the class $\mathcal{K}$ if $Q \in C[0,R^+]$, $Q(o) = o$, and $Q(r)$ is strictly monotonically increasing in r. And a function $\sigma(t)$ is said to belong to the class $\mathcal{L}$ if $\sigma \in C[J,R^+]$, is monotonically decreasing in t, and $\sigma(t) \to 0$ as $t \to \infty$.

Definition 3. The zero solution of (1) is said to be uniformly Lipschitz stable if there exist constants $M \geq 1$ and $\delta > 0$ such that

$$|x(t,t_o,x_o)| \leq M\,|x_o| \text{ for } |x_o| < \delta \text{ and } t \geq t_o \geq o.$$

The constant M is called Lipschitz constant. As usual we define the derivative

$$V'(t,x) = \limsup_{h \to o} 1/h\,[V(t+h,\, x + hf(t,x)) - V(t,x)]$$

for $(t,x) \in J x S_\rho$.

It is well known that (see [2], [7]) if $V(t,x)$ is Lipschitzian and $x(t)$ is a solution of (1), then the above definition is equivalent to:

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$$V'(t,x) = \limsup_{h \to o} 1/h\,[(V(t+h, x(t+h)) - V(t,x)]$$

Definition 4. A function $V(t,X) \geq 0$ is said to be decrescent if there exists a function $b(r) \in \mathcal{X}$ such that

$V(t,X) \leq b(\|X\|)$, $(t,X) \in JxS_\rho$.

In the sequal we need the follwing hypotheses

(H_1) $V \in C[JxS_\rho, R^+]$, $V(t,0) = 0$ and $V(t,X)$ is positive definite;

(H_2) $V'(t,X) \leq 0$;

(H_3) $V'(t,X) \leq -\phi(\|X\|)$, $\phi \in \mathcal{X}$;

(H_4) $\|X(t)\| \leq V(t,X) \leq M\|X(t)\|$, M is a constant, $M > 0$.

(H_5) V is locally Lipschitzian, i.e. for $M > 0$;

$\|V(t,X_1) - V(t,X_2)\| < M\|X_1 - X_2\|$, (t,X_1);

$(t,X_2) \in J\,xS_\rho$;

(H_6) $V(t,X)$ is a decrescent function;

(H_7) f is Lipschitzian

The following two Lemmas of [7] are needed:

<u>LEMMA 1.</u> The zero solution of (1) is uniformly asymptotically stable if and only if there exist functions $C \in \mathcal{X}$ and

$\sigma \in \mathcal{R}$ such that

$$\|X(t)\| \leq C(\|X(t_1)\|)\,\sigma(t-t_1), \quad t \geq t_1, \qquad (3.1)$$

whenever $t_1 > t_o$ and $\|X(t_1)\| < \rho$.

LEMMA 2. Suppose the zero solution of (1) is uniformly asymptotically stable. and f(t,X) which satisfies the Lipschitz condition

$$\|f(t,X_1) - f(t,X_2)\| \leq L(t)\,\|X_1-X_2\|$$

for $(t,X_1), (t,X_2) \in J x S_\rho$,
where $L(t) \geq 0$ is continuous on J and

$$\left\| \int_t^{t+m} L(u)du \right\| \leq E_1 \|m\| ,$$

E_1 is positive constant. Then, there exists a function V(t,X) satisfies the conditions (H_1), (H_3), (H_5), and (H_6)

We state the following result of [2] without proof.

Theorem 3.1 : Suppose that the hypothesis (H_7) hold. Then the zero solution of (1) is uniformly Lipschitz stable if and only if there exists a continuous function V(t,X)

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defined for $t \geq 0$ and $\| X \| < \delta$, such that

(1) $\| X \| \leq V(t,X) \leq L_1 \| X \|$, for some constant L_1

(2) $\| V(t,X_1) - V(t,X_2) \| \leq M \| X_1 - X_2 \|$,

for all $t \geq 0$, $X_1, X_2 \in R^n$, with $\| X_1 \| < \delta$, $\| X_2 \| < \delta$,

(3) $V'(t,X) \leq 0$.

Theorem 3.2. If the zero solution of (1) is unfiromly asymptotically stable, then there exists a constant $M > 0$ such that

$$\| X(t) \| \leq V(t,X) \leq M \| X(t) \| . \qquad (3.2)$$

Proof. Since the zero solution of (1) is uniformly stable, and by Lemma 1, there exist functions $C \in \mathcal{K}$ and $\sigma \in \mathcal{R}$ such that

$$\| X(t) \| \leq C(\| X(t_1) \|)\sigma(t-t_1), \; t \geq t_1 . \qquad (3.1)$$

Thus, choosing σ and C such that

$$\sigma(t) = e^{-\alpha(t-t_1)} , \qquad \alpha \text{ is nonnegative constant}$$

and

$$C(\| X(t_1) \|) = M \| X(t_1) \| , \; M \text{ is a constant, } M > 1.$$

Then, it follows that

$$\| X(t) \| \leq M \| X(t_1) \| \ e^{-\alpha (t-t_1)}, \ t \geq t_1 \qquad (3.3)$$

Moreover, if the scalar function V(t,X) is chosen such that

$$V(t,X) = \sup_{t \geq t_1} \| X(t) \| \ e^{-\alpha (t-t_1)},$$

Then,

$$\| X(t) \| \leq \sup_{t>0} \| X(t) \| = V(t,X). \qquad (3.4)$$

Thus, by (3.1) we have :

$$\| X(t) \| \leq V(t,X) \leq M \| X(t) \| .$$

<u>Theorem 3.3</u> : Let the hypothesis (H_7) be satisfied as in Lemma 2. If the zero solution of (1) is uniformly asymptotically stable, then it is uniformly Lipschitz stable.

Proof. Since the zero solution of (1) is uniformly asymptotically stable, then by Lemma 2, the function V(t,x) satisfies (H_3) and (H_5).

Moreover, by Theorem 3.2, V(t,x) satisfies (H_4). Thus the three conditions of Theorem 3.1 are satisfied and hence the proof is completed.

Remark : Theorem 3.3 is inconsistent with the conjecture of [2].

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* تعميم داله ليبنيوف لاستقرار الحلول المنتظمة للمعادلات التفاضلية العادية *

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+ قسم الرياضيات – كلية العلوم – جامعة بنها

يهتم البحث بدراسة الاستقرار التقاربى المنتظم (Uniformly asymptotically stable) للحل الصفرى للنظام غير المعتمد على الزمن (Nonautonomous system)

$$X' = f\ (t,X)$$

باستخدام داله ليبنيوف(Liapunov function) بتقليل الشروط على مشتقه دالة ليبنوف ، كما يدرس الشروط الكافية لجعل الحل الصفرى لهذا النظام يحقق الاستقرار التقاربى المنتظم (Uniformly asymptotically stable) واستقرار ليبشتز المنتظم (Uniformly Lipschitz stable) فى آن واحد.