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ON A MODIFIED HYBRID APPROACH FOR SOLVING VECTOR OPTIMIZATION PROBLEMS

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ABSTRACT

This paper presents an approach for solving vector optimization problems (VOP) which combines the characteristics of both the weghted sum norm method and constraint method. It is shown in the paper that the noninferior solutions of Vop can be characterized in terms of the optimal solutions of the presented approach. Also, in this paper, the basic notions in parametric convex programming are redefined and analyzed qualitatively for VOP. Such analysis gives us the possibility of relating different SOP's with each other.

INTRODUCTION

In an earlier work Chankong and Háimes gave necessary and sufficient conditions for the determination of the noninferior solutions for vector optimization problems (VOP) using the hybrid approach which combines the characteristics of both the nonnegative weighted sum method and the k th - objective ε - constraint method [1]. Furthermore, they determined the necessary and sufficient conditions for the determination of the noninferior solutions for VOP using the weighted sum norm method. Also, Osman in [3] introduced and basic notions in parametric convex programming problems.

This paper is devoted to characterize the noninferior solutions for VOP using the modified hybrid approach which combines the characteristics of both the weighted sum norm method and the k th-objective $-\varepsilon-$ constraint method. The basic notions in parametric convex programming are redefined and analyzed qualitatively for VOP in this paper. Also, the paper presents several results which relate two convex programs with each other.

Problem formulation .

Let us consider the following vector optimization problems:

$$\min \left(f_1(x), f_2(x), \ldots, f_n(x)\right)$$

VOP subject to

$$X = \left\{ x \in R^N \mid g_i(x) \leqq o, \quad i=1,2,\ldots,m \right\}$$

Let us define the following scalarization of VOP.

This approach combines the characteristics of both the weightednorm and constraint characterization. Accordingly, it will be called the hybrid characterization. The formulation of this approach can be stated as :

$$\min_{x \in X} \sum_{j=1}^{n} w_j \left| f_j(x) - f_j^* \right|^P$$

P(w , p, ε)

subject to $f_j(x) \leq \varepsilon_j$ for all $j = 1,2, \ldots,n$,

where $1 \leq p < \infty$, $f_j^* = \min_{x \in X} f_j(x)$ and the w_j's are nonnegative weights satisfying $w_j > \hat{o}$ for all $j=1,2,\ldots,n$ and $\sum_{j=1}^{n} w_j = 1$. Here noninferier solutions of VOP are characterized in terms of P(w , p, ε).

Characterization of the noninferior solutions.

The following theorems establish the relation between the noninferior solutions of VOP and the optimal soutions of P(w , p , ε).

Theorem 1 x^* is noninferior solution of VOP if and only if x^* is an optimal solution of P(w, p, ε) for any given $w_j > o$, $j = 1,2,\ldots,n$ and for some $\varepsilon \in R^n$ for which P(w, p, ε) is feasible.

Proof. Necessity-assume that for any given $w_j^\circ > o$, $j = 1,2,\ldots,n$, x^* does not solve P(w° , p , ε) for any ε including P(w°, p , ε^*), where $\varepsilon_j^* = f_j(x^*)$, $j = 1,2,\ldots,n$. Let x° be an optimal solution of P(w°,p,ε^*). Hence we have :

$$\sum_{j=1}^{n} w_j^\circ \left| f(x^\circ) - f_j^* \right|^P < \sum_{j=1}^{n} w_j^\circ \left| f_j(x^*) - f_j^* \right|^P$$

and $f_j(x^\circ) \leqq f_j(x^*)$ for all $j = 1,2,...,n$. Hence x^* is not noninforior solution proving necessity.

Sufficiency-suppose that x^* solves $P(w^\circ, p, \varepsilon)$ for some $\varepsilon \in R^n$. It must also solve $P(w^\circ, p, \varepsilon^*)$, where $\varepsilon_j^* = f_j(x^*)$, $j = 1, 2, ... ,n$. Suppose x^* is not noninferior solution of VOP, then there exists $x^\circ \in X$ such that :

$f_j(x^\circ) \leqq f_j(x^*)$ for all $j= 1,2,.....n$, then

$$\sum_{j=1}^{n} w_j^\circ |f_j(x^\circ) - f_j^*|^P \leqq \sum_{j=1}^{n} w_j^\circ |f_j(x^*) - f_j^*|^P,$$

$w^\circ_j > o$, $j = 1,2,..... , n$.

This clearly contradicts the assumption that x^* solves $P(w^\circ, P, \varepsilon^*)$. Since $x^\circ \in X$, then x^* must be noninferior solution of VOP.

Theorem 2 Assume that one of the following holds:

(i) $P(w^\circ, P, \varepsilon)$ is stable, X is convex set , and $f_j(x)$, $j = 1,2,...,n$ are convex functions throughout R^N, or

(ii) all $f_j(x)$, $j=1,2,..., n$ and $g_i(x)$, $i=1,2,...,m$ are faithfully convex (all these functions are either linear or nonlinear and contain no stright-line segment in their graphs) throughout R^N and X, then :

$$X^* = \phi \quad \text{if} \quad \Psi(\varepsilon^*) = -\infty ,$$

where X^* is the set of all noninferior solutions of VOP.

Proof. Since

$$\Psi(\varepsilon^*) = \inf \left\{ \sum_{j=1}^{n} w_j^{\circ} |f_j(x) - f_j^*|^P \middle| x \in X \text{ and } f_j(x) < \varepsilon_j^*, \; j = 1,2, \ldots,n \right\}.$$

$$= -\infty$$

The weak duality theorem requires that the dual of $P(w^\circ, p, \varepsilon^*)$ also be $-\infty$, that is

$$\sup_{u \geq 0} \left[\inf_{x \in X} \left\{ \sum_{j=1}^{n} w_j^\circ |f_j(x) - f_j^*|^P + \sum_{j=1}^{n} u_j (f_j(x) - f_j(x^\circ)) \right\} \right] = -\infty$$

Suppose that there exists a noninferior solution say x° of VOP. From Theorem 1, x° solves $P(w^\circ, p, \varepsilon^\circ)$ where $f_j(x^\circ) = \varepsilon_j^\circ$, $j = 1, 2, \ldots, n$ If assumption (i) prevails, apply Geoffrion's strong duality theorem, and if assumption (ii) prevails, apply

$$\sum_{j=1}^{n} w_j^\circ |f_j(x^\circ) - f_j^*|^P$$

$$= \sup_{u \geq 0} \left[\min_{x \in X} \left\{ \sum_{j=1}^{n} w_j^\circ |f_j(x) - f_j^*|^P + \sum_{j=1}^{n} u_j (f_j(x) - f_j(x^\circ)) \right\} \right].$$

In any case we have

$$-\infty < \sum_{j=1}^{n} w_j^\circ |f_j(x^\circ) - f_j^*|^P$$

$$= \sup_{u \geqq 0} \left[\inf_{x \in X} \left\{ \sum_{j=1}^{n} w_j^{\circ} |f_j(x) - f_j^*|^P + \sum_{j=1}^{n} u_j(x) - f_j(x^{\circ}) \right\} \right]$$

which is a contradiction. Thus X^* must be empty.

The set feasible parameters

Definition 1. The set of feasible parameters for problem $P(w,p,\epsilon)$, denoted by U, is defined by :

$$U = \{ \varepsilon \in R^n \mid X(\varepsilon) \neq \phi \} \quad ,$$

where $X(\varepsilon) = \{ x \in R^N \mid x \in X \text{ and } f_j(x) \leqq \varepsilon_j \, , \, j=1,2,\ldots,n \}$

Theorem 3. If VOP is convex , then the set U is convex .

Proof. Let $\varepsilon^1, \varepsilon^2 \in U$ and $\varepsilon^1 \neq \varepsilon^2$, then there exist points x^1 , x^2 in R^N respectively such that $f_j(x^1) \leqq \varepsilon_j^1$ and $f_j(x^2) \leqq \varepsilon_j^2$, $j= 1,2, \ldots,n$. Therefore $(1-\alpha)f_j(x^1) + \alpha f_j(x^2) \leqq (1-\alpha)\varepsilon_j^1 + \alpha\varepsilon_j^2$, $j= 1,2,\ldots,n$ for all $0 \leqq \alpha \leqq 1$. From the convexity of the functions $f_j(x)$, $j = 1,2,\ldots,n$ and the set X it follows that :

$f_j[(1-\alpha)x^1 + \alpha x^2] \leqq (1-\alpha)\varepsilon_j^1 + \alpha\varepsilon_j^2$, $j=1,2, \ldots, n$ and $(1-\alpha)x^1 + \alpha x^2 \in X$. Then $X[(1-\alpha)\varepsilon^1 + \alpha\varepsilon^2] \neq \phi$.

This means that $(1-\alpha)\varepsilon^1 + \alpha\varepsilon^2 \in U$ for all $0 \leqq \alpha \leqq 1$.
Hence the set U is cinvex .

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Theorem 4. If there is $\varepsilon \in U$ such that $X(\varepsilon)$ is bounded, then the set U is closed.

Proof. Suppose that $\bar{\varepsilon} \in R^n$ is a frontier point of U, then any neighbourbood of $\bar{\varepsilon}$ has a nonempty intersection with U, it follows that $\bar{\varepsilon} + \nu \in U$ for any $\nu > o$. Consider the sequence $\{\bar{\varepsilon} + \nu^n\}$; $\nu^{n+1} < \nu^n$; $\nu^n \to o$ as $n \to \infty$ $(n = 1,2, \ldots.)$. The set $X(\bar{\varepsilon}, \nu^n)$ is compact since it is closed and bounded, with

$$X(\bar{\varepsilon} + \nu^{n+1}) \subseteq X(\bar{\varepsilon} + \nu^n), \qquad n+ 1,2,\ldots\ldots$$

Therefore, it flollows that :

$$\bigcap_{n=1}^{\infty} X(\bar{\varepsilon} + \nu^n) = X(\bar{\varepsilon}) \neq \phi$$

and hence the result .

The solvability set .

Definition 2. The solvability set of $P(w, p, \varepsilon)$, denoted by B, is defined by :

$$B = \{(w, \varepsilon) \in R^{2n} \mid X^*(\varepsilon) \neq \phi\},$$

where $X^*(\varepsilon)$ is the set of all optimal solutions of $P(w, P, \varepsilon)$.

Let us decompose the set B into B_1 and B_2 , where B_1 is the orthogonal projection of B on ε - space, that is the orthogonal projection of B on ε -space , that is

$$B_1 = \{ \varepsilon \in R^n \mid (w , \varepsilon) \in B \} .$$

Also, B_2 is the orthogonal projection of B on w-space, that is

$$B_2 = \{ w \in R^n \mid (w , \varepsilon) \in B \} .$$

<u>Theorem 5.</u> If for one $\varepsilon \in B_1$, it holds that $X^*(\varepsilon)$ is bounded, then $B_1 = U$.

<u>Proof.</u> Suppose that $\varepsilon^*_J = f_j (x^*)$, $j = 1,2,\ldots, n$, where x^* is an optimal solution of $P(w , p , \varepsilon)$ it follows from the assumption that

$$X^*(\varepsilon^*) = \{ x \in R^N \mid x \in X \text{ and } f_j (x) \leqq \varepsilon^*_j , \quad j = 1, 2, \ldots, n. \}$$

is bounded. Hence the set

$$X^*(\varepsilon) = \{ x \in R^N \mid x \in X \quad \text{and } f_j(x) < \varepsilon_j , j=1,2, \ldots n. \}$$

is bounded, see [3[, for all $\varepsilon \in R^n$ for which $X^*(\varepsilon) \neq o$.
It follows from Theorem 3 and Theorem 4 that the set U is

unbounded, convex and closed set. Suppose $\varepsilon^{o} \in U$, then $X^{*}(\varepsilon^{o}) \neq o$, therefore, there exists $x^{o} \in X^{*}(\varepsilon)$, which implies $U \subseteq B_1$, and hence $B_1 = U$.

It is clear that the set $B_2 \cup \{o\}$ is a cone with vertex at the origin.

The stability set of the first kind.

Definition 3. Assume that the problem $P(w, p, \varepsilon)$ is stable for (w^{*}, ε^{*}) with a corresponding noninferior solution x^{*}, then the stability set first kind corresponding to x^{*}, denoted by $S(x^{*})$, is defined by

$$S(x^{*}) = \Big\{ (w, \varepsilon) \in R^{2n} \Big| \sum_{j=1}^{n} w_j \, |f_j(x^{*}) - f_j^{*}|^{P} = \min_{x \in X} \sum_{j=1}^{n} |w_j \; f_j(x) - f_j^{*}|^{P} \Big\}.$$

By decomposing the set $S(x^{*})$ into $S_1(x^{*})$ and $S_2(x^{*})$, where $S_1(x^{*})$ is the orthogonal projection of $S(x^{*})$ on

ε- space, that is

$$S_1(x^*) = \{ \varepsilon \in R^n \mid (w, \varepsilon) \in S(x^*) \},$$

and $S_2(x^*)$ is the orthogonal projection of $S(x^*)$ on w-space, that is

$$S_2(x^*) = \{ w \in R^n \mid (w, \varepsilon) \in S(x^*) \}.$$

Theorem 6. The set $S_1(x^*)$ is star shaped and closed. The proof is similar to the proof which was given in [3]. Also it is clear that :

(1) the set $S_2(x^*)$ is convex

(2) $S_2(x^*) \cup \{o\}$ is a closed convex cone with vertex at the origin, and

(3) if interior $[S_2(\bar{x}) \cap S_2(x^*)] \neq \phi$, then $S_2(\bar{x}) = S_2(x^*)$

Related parametric convex programs.

In this section several results will be presented which relate the convex programming problem $P(w, p, \varepsilon)$ with parameters in both objective function and the right hand side of the constraints to the convex programming problem $P(w, \varepsilon)$ with parameters in both objective function and the right hand side of the constraints.

It is well known from the literature that noninferior solutions of VOP are characterized in terms of the optimal solution of problem $P(w, \varepsilon)$.

At a point x^*, the Kuhn-Tucker conditions of $P(w, p, \varepsilon)$ take the form, see [2] ,

$$\sum_{j=1}^{n} pw_j \left| f_j(x^*) - f_j^* \right|^{p-1} \frac{\partial f_j(x^*)}{\partial x_\alpha} + \sum_{j=1}^{n} \mu_j \frac{\partial f_j(x^*)}{\partial x_\alpha}$$

$$+ \sum_{i=1}^{n} \nu_i \frac{\partial g_i(x^*)}{\partial x_\alpha} = 0, \quad \alpha = 1,2,\ldots\ldots N,$$

$$f_j(x^*) - \varepsilon_j^* \leqq o, \qquad j = 1,2,\ldots\ldots,n,$$

$$g_i(x^*) \leqq o, \qquad i = 1,2,\ldots\ldots,m,$$

$$\mu_j(f_j(x^*) - \varepsilon_j^*) = o, \qquad j = 1,2,\ldots\ldots,n,$$

$$\nu_i \; g_i(x^*) = o, \qquad i = 1,2,\ldots\ldots m,$$

$$\mu_j \geqq o, \quad j = 1,2, \ldots n \quad \text{and} \quad \nu_i \geqq o, \quad i=1,2,\ldots,m$$

Also, the Kuhn-Tuker conditions of problem $P(w, \varepsilon)$ take the form :

$$\sum_{j=1}^{n} w_j \frac{\partial f_j(x^*)}{\partial x_\alpha} + \sum_{j=1}^{n} \bar{\mu}_j \frac{\partial f_j(x^*)}{\partial x_\alpha} + \sum_{i=1}^{m} \bar{\nu}_i \frac{\partial g_i(x^*)}{\partial x_\alpha} = o,$$

$$\alpha = 1,2, \ldots,N,$$

$$f_j(x^*) - \varepsilon_j^* \leqq o \ , \ j=1,2,\dots,n,$$

$$g_i(x^*) \leqq o \ , \ i=1,2,\dots,m,$$

$$\bar{\mu}_j(f_j(x^*) - \varepsilon_j^*) = o \ , \ j=1,2,\dots,n,$$

$$\bar{\nu}_i \ g_i(x^*) = o \ , \ i=1,2,\dots,m,$$

$\bar{\mu}_j \geqq o$, $j= 1,2,\dots,n$ and $\bar{\nu}_i \geqq o$, $i= 1,2,\dots,m$.

Theorem 7. If x^* is an optimal solution of $P(w , P, \varepsilon)$ $(w, P, \varepsilon) = (w^*, P^*, \varepsilon^*)$ and $(x^*, w^*, P^*, \varepsilon^*, \mu^*, \nu^*)$ solves the Kuhn-Tucker conditions of $P(w , P , \varepsilon)$, then x^* is an optimal solution of $P(w , \varepsilon)$ for $w_j = P^* w_j^* |f_j(X^*) - f_j^*|^{P^*-1}$, $j=1,2,\dots,n$. And if $\bar{x}$ is an optimal solution of $P(w , \varepsilon)$ for $(w, \varepsilon) = (\bar{w} , \bar{\varepsilon})$ and $(\bar{x} , \bar{w} , \bar{\varepsilon} , \bar{\mu}, \bar{\nu})$ solves the Kuhn-Tucker conditions of $P(w , \varepsilon)$, then $\bar{x}$ is an optimal solution of $P(w , p , \varepsilon)$ for $Pw_j |f_j(\bar{x})-f_j^*|^{p-1} = \bar{w}_j$, $j = 1,2, \dots, n$.

The proof follows directly from the Kuhn-Tucker conditions of problems $P(w , p , \varepsilon)$ and $P(w , \varepsilon)$.

Theorem 8. If x^* is an optimal solution of the problem $P(w , p , \varepsilon)$ and $(x^*, w^*, P^*, \mu^*, \nu^*)$ solves the Kuhn-Tucker conditions of problem $P(w , p , \varepsilon)$ with $w^* > o$ and

$f_j(x)$, $j = 1,2, \ldots,n$ are strictly convex functions on R^n , then x^* is a noninferior solution of VOP.

The proof is clear.

Example Consider the following VOP :

$$\min \left((x-1)^2 + y,\ \ x + y\right)$$

$$\text{subject to } X = \{(x , y) \in R^2 \mid x , y > o\}$$

It is clear that $(f_1^* , f_2^*) = (o , o)$, therefore

$$\min \left(w_1 |(x-1)^2 + y|^P + w_2 |x + y|^P\right)$$

$P(w , p , \varepsilon)$:

$$\text{subject to } X(\varepsilon) = \{(x,y) \in R^2 \mid x,y \geqq o,\ (x-1)^2 + y \leqq \varepsilon_1 \text{ and } x + y \leqq \varepsilon_2\},$$

where $1 \leqq p < \infty$, ,w_1 , $w_2 \geqq o$ and $w_1 + w_2 = 1$.

The following sets are obtained :

$$U = \{\varepsilon \in R^2 \mid \varepsilon_1 \geqq o ,\ \varepsilon_2 > o \text{ and } \varepsilon_1 \geqq \varepsilon_2 - \tfrac{5}{4}\} = B_1$$

It is clear that the point $(\frac{1}{2} , o)$ is a noninferior of VOP. Let $P = 1$, then

$$S_1(\tfrac{1}{2} , o) = \{\varepsilon \in R^2 \mid \varepsilon_1 \geqq \tfrac{1}{4} \text{ and } \varepsilon_2 \geqq \tfrac{1}{2}\}, \text{ and}$$

$$S_2(\tfrac{1}{2} , o) = \{w \in R^2 \mid w_1 > o ,\ w_2 > o \text{ and } w_1 = w_2\}.$$

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مدخل معدل باستخدام المزج لحل المشاكل الأمثلية الأتجاهيـة

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هذا البحث أوجد مدخل باستخدام مزيج من الاوزان المعيارية ومدخل القيود ، وقد أظهر هذا البحث أن الحلول الأكثر كفاءة يمكن تمييزها بدلالة الحل المثالى الخاص بهذا المدخل ، كما أعيد تعريف وتحليل توصيفى لمفاهيم أساسية للبرمجة البارامترية المحدبة من أجل المشاكل الأمثلية الاتجاهية ، وقد أوجد هذا البحث العلاقة بين هذا المدخل وأحد المداخل المعروفة سابقا ٠