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ISOCHRONAL SURFACES FOR CERTAIN CONTROL SYSTEM

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ABSTRACT

For a class of multi-input multi--output contrl systems, the problem of determining the isochronal surfaces will be considerd in this work. For a special class of control systems, it is shown that the trajectories which remain on the boundary of this isochronal surface are asymptotically stable under the action of the control law which generates them. The isochronal surfaces are constructed using a method based on the construction of Lyapunof like function.

1. INTRODUCTION

Consider an object of control whose motion in the phase space $X \subset R^n$ of the variables $x_1, x_2, \ldots\ldots\ldots, x_n$ is described by the linear differential system

$$\dot{X} = A \quad X + B \quad U , \tag{1.1}$$

where $U = (u_1, u_2, \ldots\ldots\ldots\ldots, u_r)$ is the r-column vector of the controlling actions, A and B are two constant matrices of appropriate dimensions. The domain of control is a convex and compact set, $\Omega \subset R^r$, and can be considerd as a unit cube

$$\Omega = \{ U \mid |u_j| \leqslant 1, \quad j = 1,2,\ldots\ldots,r \} \qquad (1.2)$$

in the r-dimensional space R^r.

Throughtout this work, system (1.1) is assumed to be operating time optimally, i.e. the control input $u_j(t)$, $t_0 \leqslant t \leqslant t_f$, is such that the system is transferred from an initial state $X(t_0) = (x_{10}, x_{20}, \ldots\ldots\ldots, x_{n0})$ at $t=t_0$ to the origin of the state space in a minimal time t^*. The isochronal surface can be defined as the boundary of some sets, called the set of attainability. In a different context, these sets have been used to characterize the region of controllability with capture to the target and to increasing the effect in the stability of an ecosystem against external perturbations. Several techiques employ the Pontryajin maximum principle in its abnormal form, have been used to find or to approximate this set of attainability. One of these methods is based upon the hypothesis that if a trajectory is on the isochronal surface, then the reacability maximum principle provides a necessary condition that must be satisfied by this boundary trajectory. The resulting trajectories will approach the isochronal surface assymptotically. This method is readily applied to low dimensional systems having a single control parameter, [2],. The assymptotic stability of this boundary trajectories under the action of the abnormal control has been proven in general.

An alternative approach is to use a Lyapunof method to estimate the isochronal surface. This method also has been applied to a control system which admits a scalar control, [4],.

The present work is concerned with the determination of the isochronal surface for certain multi-input multi-output optimal control systems. A suitable Lyapunof like function will be constructed to allow the determination of the set of attainability. It will be shown that for the control system (1.1) the abnormal control will drive the system asymptotically to the considerd isochronal surface. Advanteges of this method are that it is applicable to higher dimensional systems and does not require calculation of orbits or a choice of boundary conditions. It will be shown that the trajectories which remain on the boundary of isochronal surface must be asymptotically stable under the action of the control law which generates them.

2. THE OPTIMAL CONTROL SYSTEM

Restrictingattention to system (1.1), some useful properties of optimal control systems relevent to the enusing analysis are summarized below .

1- The boundary of the set of attainability formes a continuous surface, the minimum isochronal surface, in the state space and is composed of the set of states from which the origin can be reached in optimal time t^*.

2- The isochronal surface form a family of closed convex surface which expand monotonically from the state origin as t^* increases on the open interval $(0, \infty)$.

3- If the eigenvalues λ_i are real and non-positive, then control system (1.1) is a time optimally controllable to the state origin from all states X of the state space.

4- If λ_i are real and positive, system (1.1) is not time optimally controllable but it exhibites a bounded region of controllability.

5- The time optimal control, if it exists, is unique and piecewise constant with at most (n-1) switches for each control parameter, these switches occurs at the vertices v_h of Ω.

6- If any of λ_i is complex, then the number of switching points is finite and depends on the distance from the initial states to the target set, the origin.

7- Equilibrium points of controlled system may or may not lie in the isochronal surface.

8- Optimal trajectories may approach the isochronal surface asymptotically from inside the set of attainability, or

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it can not cross and meet these surfaces, or it may lie entirely in isochronal surfaces.

This last property provides the basis of the analysis used in this work. A method for calculating a like Lyapunov function which can be used to construct the set of attainability will be present in the following section.

3. LYAPUNOV LIKE FUNCTION

Consider control system (1.1). Assume that this system is asymptotic stable, each of λ_i has a negative real part, then this system possess a bounded non empty attainability set from the origin. Summer's results, [4], canbe generalised in the context of the present work as follows.

Let V be a continuously differentiable function
$V / R^n \longrightarrow R$
such that

i- $V(0) = 0$;

ii- $\{ X \in R^n / V(X) \leqslant V^* \} \subset \{ X \in R^n \mid V(X) \leqslant W \}$ for all values of W such that $W > V^*$, where V^* is some specified positive real number;

iii- $\{ X \in R^n \mid \dot{V} (X,U) \geqslant 0 \} \subset \{ X \in R^n \mid V(X) \leqslant V^* \}$ for all u such that $|u_j(t)| \in \Omega$.

Then the set of attainability S from the origin for control system under consideration is formed from the union of subsets S_j safisfy

$$S_j \subseteq \{X \in R^n \mid V(X) \leqslant V_j^*\} \tag{3.1}$$

Let V(X,U), a Lyapunov like function, be a quadratic form

$$V(X) = X^T P X \tag{3.2}$$

where P is a $n \times n$ real symmetric matrix to be found.

Condition i) is satisfied. For condition ii), V must be positive difinite. Lyapunov like function, V = constant, represents a family of hyper-ellipsoids with centers at the origin

Using equations (1.1) and (3.2), we obtain

$$\dot{V}(X,U) = (\nabla \;\; V)^T.(AX + BU) \tag{3.3}$$

$$= - X^T Q X + R$$

where

$$Q = -\ (PA + A^T P)\ ; \tag{3.4}$$

$$R = X^T\ PBU + U^T B^T\ PX \tag{3.5}$$

Condition iii requires that

$$X^T QX - R \leqslant 0 \tag{3.6}$$

Seting Q = I, I is the identity matrix, then equation (3.6) represents the interior and the boundary of a hypersphere passing throught the origin.

In order to obtain V^*, one must maximize V(X) subject to condition (3.6). V_j^* is equal to this maximum values corresponding to each vertex of the control domain. The required set of attainability is the union of sets which can be estimated by using equation (3.1). Assuming that the control domain has a finite number of vertices k. Since this control domain is symmetric with respect to the origin, then the number of maximal points X_{hM}, h= 1,2, ...,k: will be reduced to k/2 maximal points.

Using Lagrangian method to obtain these maximal points. Lagrangian function takes the following form

$$\phi(X,U,\alpha\ ) = V(X) + \psi\,(X,U) \tag{3.7}$$

where $\psi(X^T U) = X^T QX + R,$ (3.8)

and α is the Lagrangian multipliers. The required necessary conditions for maximization of V takes the form

$$\frac{\partial V/\partial x_1}{\partial \psi/\partial x_1} = \frac{\partial V/\partial x_2}{\partial \psi/\partial x_2} = \ldots\ldots\ldots = \frac{\partial V/\partial x_n}{\partial \psi/\partial x_n} = (-\alpha) \qquad (3.9)$$

Equations (3.9) and (3.8) constitute n quadratic equations necessary to determine the n-components of the corresponding maximal points.

In the following section, three two dimensional control systems are considered. The first one is concerned with real eigenvalues and double control parameters. In the second example complex eigenvalues and a scalar control problem will be considered. The last one is concerned with complex eigenvalues and two control parameters.

4. EXAMPLES

Example 1.

Consider the control system

$$\begin{aligned} \dot{x} &= -2x + u_1 + u_2 \\ \dot{y} &= -y + u_1 \end{aligned} \qquad (4.1)$$

where $X = (x,y)^T$ and $|u_1| \leqslant 1$, $|u_2| \leqslant 1$, the control domain is the unit symmetric square with vertices v_h, h=1,2,3,4. Eigenvalues of the undisturbed system are -1, and -2. Let V(x,y) to be

$$V(x,y) = p_{11}x^2 + 2p_{12}xy + p_{22}y^2 \tag{4.2}$$

$$\dot{V}(X,U) = -(4p_{11}x^2 + 6p_{12}xy + 2p_{22}y^2 - 2(p_{11}x + p_{12}(x+y) + p_{22}y)u_1 - 2(p_{11}x + p_{12}y)u\ ). \tag{4.3}$$

Using equation (3.4) one can determin p_{ij} as follows

$$p_{11} = 1/4, \qquad p_{12} = 0, \qquad p_{22} = \tfrac{1}{2} \tag{4.4}$$

From equations (4.3,4)

$$\{(x,y) \mid \dot{V}(X,U) \geqslant 0\} = \bigcup_{u_1,u_2=v_h} \{(X,U) \mid (x-\tfrac{1}{4}(u_1 + u_2))^2 + (y-\tfrac{1}{2}u_1)^2 \leqslant 1/16(5u_1^2+2u_1u_{22}+u_2^2)\} \tag{4.5}$$

to get S_j one must maximize

$$V(x,y) = \tfrac{1}{4}x^2 + \tfrac{1}{2}y^2 \tag{4.6}$$

subject to either

$$(x - \tfrac{1}{2})^2 + (y-\tfrac{1}{2})^2 \leqslant \tfrac{1}{2} \tag{4.7}$$

or
$$x^2 + (y-\tfrac{1}{2})^2 \leqslant 1 \tag{4.8}$$

The required maximal point X_M++ must satisfy

$$\tfrac{1}{2}x/(2x-1) = y/(2y-1) \tag{4.9}$$

while X_M+- must satisfy

$$\tfrac{1}{2}x/2x = y/(2y-1) \tag{4.10}$$

From equations (4.7,9) the coordinates of X_M++ must satisfies

$$4x^3 - 12x^2 + 15x - 6 = 0, \tag{4.11}$$

$$y = x/(2(1-x)) \tag{4.12}$$

The cubic equation (4.11) has at least one single real root which must be found numerically. In this case the coordinates of X_M++ are given by

$$x = 0.702, \qquad y = 1.177, \tag{4.13}$$

and so $V_1^* = V_{++}^* = 0.8159$ and the corresponding S_1 takes the form

$$S_1 \subseteq \{(x,y) \in R^2 \mid x^2 + 2y^2 \leqslant 3.264\} \tag{4.14}$$

From equations (4.8,10) the coordinates of X_M+- are given by

$$x = 0.97, \quad y = -0.5$$

So the corresponding S_2 takes the form

$$S_2 \subseteq \{(x,y) \in R^2 \mid x^2 + 2y^2 \leqslant 1.44\} \tag{4.15}$$

The required set of attainability in this case is formed from the union of S_1 and S_2.

<u>Example 2.</u>

Consider the control system

$$\dot{x} = y \tag{4.16}$$

$$\dot{y} = -x - y + u$$

where $|u| \leqslant 1$, and the eigenvalues of system matrix are $(-1 \pm \sqrt{3}i)/2)$. The Lyapunove function takes the form

$$V(x,y) = p_{11}x^2 + 2p_{12}xy + p_{22}y^2 \tag{4.17}$$

where

$$\dot{V} = -(2p_{12}x^2 + 2(p_{22} + p_{12} - p_{11})xy + 2(p_{22}-p_{12})y^2 -$$

$$-2p_{22}uy - 2p_{12}ux) \qquad (4.18)$$

where $p_{11} = 3/2$, $p_{12} = \frac{1}{2}$ and $p_{22} = 1$. (4.19)

from equations (4.18,19)

$$\{(x,y) \mid \dot{V}(X,U) \geqslant 0\} = \{(x,y) \in R^2 \mid (x-\tfrac{1}{2}u)^2+(y-u)^2 \leqslant$$

$$(\sqrt{5}u/2)^2\} = \{(x,y) \mid (x-\tfrac{1}{2})^2 + (y-1)^2 \leqslant (\sqrt{5}/2)^2\} \cup \{(x,y) \mid$$

$$(x+\tfrac{1}{2})^2 + (y+1)^2 \leqslant (\sqrt{5}/2)^2\} \qquad (4.20)$$

maximize $V(x,y) = 3/2x^2 + xy + y^2$

subject to $(x-\frac{1}{2})^2 + (y-1)^2 = 5/4$ (4.21)

The necessary condition takes the form

$$(3x + y)/(2x-1) = (x+2y)/(2y-2) \qquad (4.22)$$

From equations (4.21,22) the coordinates of the maximal point must satisfies

$$4y^3 - 12y^2 + 17y - 14 = 0, \qquad (4.23)$$

$$x = 4y(1-y)/(2y-7).$$

The numerical solution of the cubic equation gives

$$y = 1.711,\ x = 1.36,\ \text{and so } V^{*} = 8.03. \tag{4.24}$$

The required set of attainability S must satisfy

$$S \subseteq \{(x,y)\in R^2 \mid 3x^2 + 2xy + 2y^2 \leqslant 16.06\} \tag{4.25}$$

Example 3.

Consider the control system

$$\dot{x} = y + u_1$$

$$\dot{y} = -x - y + u_2 \tag{4.26}$$

where the control domain is as in Example 1, and the eigenvalues of the undisturbed system are as those of Example 2. The Lyapunov like function takes the form

$$V(x,y)= p_{11}x^2 + 2p_{12}xy + p_{22}y^2. \tag{4.27}$$

Using the same method p_{ij} takes the following values

$$p_{11} = 3/2\ ,\ p_{12} = \tfrac{1}{2}\ ,\ \text{and } p_{22} = 1. \tag{4.28}$$

To estimate the set of attainability one must maximize

$$V(x,y) = 3/2x^2 + xy + y^2 \tag{4.29}$$

subject to either

$$(x-2)^2 + (y-3/2)^2 = (5/2)^2 \tag{4.30}$$

or $$(x-1)^2 + (y+\tfrac{1}{2})^2 = (\sqrt{5}/2)^2 \tag{4.31}$$

By using equation (3.8) the coordinates of the maximal point X_M++ must satisfy the equations

$$20y^3 - 4y^2 + 526y - 559 = 0 ,$$

$$x = y(1-4y) / (2y-13). \tag{4.32}$$

and the numerical solution is found to be

$$x = 0.345, \quad y = 1.098 \quad \text{and} \quad V^* = 1.764 \qquad \text{giving}$$

$$S_1 \subseteq \left\{(x,y) \in R^2 \,\middle|\, 3x^2 + 2xy + 2y^2 \leqslant 3.53 \right\} \tag{4.33}$$

The coordinates of the maximal point X_M+- must satisfy

$$5y^3 + 27y^2 + 84y + 15 = 0,$$

$$x = -y(y + 7) / (2y + 1) \tag{4.34}$$

The numerical solution is found to be

$$x = -0.863, \quad y = 0.159 \quad \text{and} \quad V^* = 1.005 \quad \text{giving}$$

$$S_2 \subseteq \{(x,y) \in R^2 \mid 3x^2 + 2xy + 2y^2 < 2.01 \tag{4.35}$$

The corresponding set of attainability must be contained in the union of S_1 and S_2 .

5. CONCLUSION

For a special class of optimal control systems it is shown that, trajectories which remain on the boundary of set of attainability must be asymptotically stable.
A method based on construction of a Lyapunov like function helps us to estimate these sets and to determine the isochronal surfaces. Different cases according to either the system matrix has real or complex eigenvalues, or the system has scaler or vector control parameters, have been considered.

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السطوح متساوية الزمن لبعض نظم التحكم

د . / محمود زكى رجب

قسم الرياضيات ـ كلية العلوم

جامعة الزقازيق

لفئة من نظم التحكم متعددة الداخل ومتعددة الخرج فأن مشكلة تحديد السطوح متساوية الزمن اخذت فى الاعتبار فى هذا البحث .

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