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SOME REMARKS ON TOTAL CHROMATIC NUMBER

By

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ABSTRACT

We determine here the total chromatic number of: The conjunction of P_n and P_m, the conjunction of P_m and C_n, the conjunction of $K_{1,n}$ and P_m, the conjunction of $K_{1,n}$ and C_m, the conjunction of $K_{1,m}$ and $K_{1,n}$, wheels, helms, webs, trees and the square of a cycle. We study the influence of elementary homeomorphism on the chromatic number of P_n, T_n and C_n. Finally we define uniquely totally coloured graphs and we show that all paths and circuits C_n , $n \equiv 0 \bmod 3$ are uniquely totally coloured.

INTRODUCTION

An element of a graph G = (V,E) is a vertex or an edge. In a total colouring two elements of G which are either adiacent or incident, must have different colours. The minimum number of colours needed for a total colouring of G is the total chromatic number $X''(G)$. We follow the notation of [7], in particular $\Delta = \Delta(G)$ is the maximum degree of the graph. The total colouring was independently introduced by Vizing [9] and Behzad [1], [2]. Both Behzad and Vizing conjectured that every graph G satisfies the following inequaliting

$$\Delta + 1 \leq X''(G) \leq \Delta + 2$$

We call graphs which need $\Delta + 1$ colours type 1 and those which need at least $\Delta + 2$ colours type 2. The lower bound of $X''(G)$ is clearly exact. An obvious upper bound is $2\Delta + 1$. There are so many interesting results concerning total chromatic numbers such as those in [3], [4], [6] while [5] represents much interesting results on edge colouring which is very important basis for total colouring.

Theorem 1

(i) $P_m \wedge P_n$ is of type 1, $(m,n) \neq (2,2)$

(ii) $P_2 \wedge C_n$ is of type 1 if and only if $n \equiv 0 \bmod 3$

(iii) $P_m \wedge C_n$, $m > 3$ is of type 1

Proof

i- For $(m,n) = (2,2)$, we have the graph $P_2 \wedge P_2$ which is isomorphic to $2P_2$, of typ 2. For $m=2$, $n \geq 3$, the graph $P_2 \wedge P_n$ is isomorphic to $2P_n$, of type 1. For $(m,n) \neq (2,2)$, this graph consists of two disjoint identical subgraphs of the cartesian product of P_m and P_n. According to [8], this is of type 1. Figure 1 shows the colouring of $P_{10} \wedge P_8$

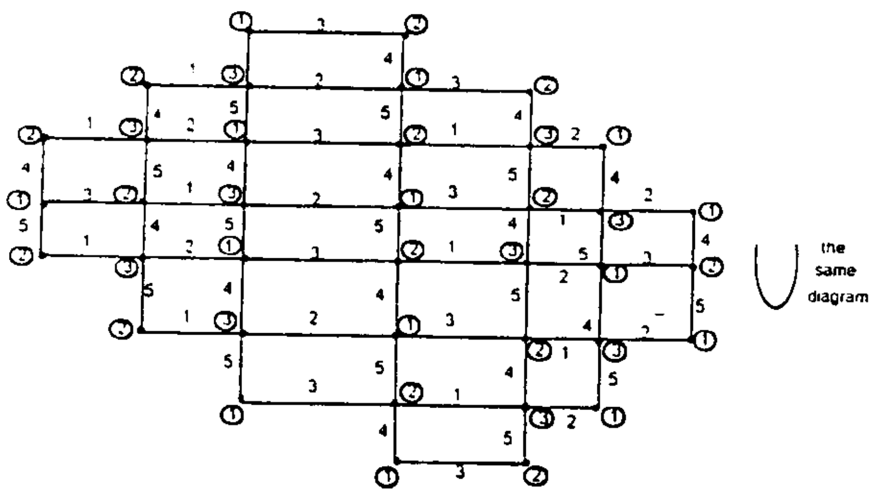


Figure 1

$$(ii)\ P_2 \wedge C_n = \begin{cases} 2C_n & : n \text{ is even} \\ C_{2n} & : n \text{ is odd} \end{cases}$$

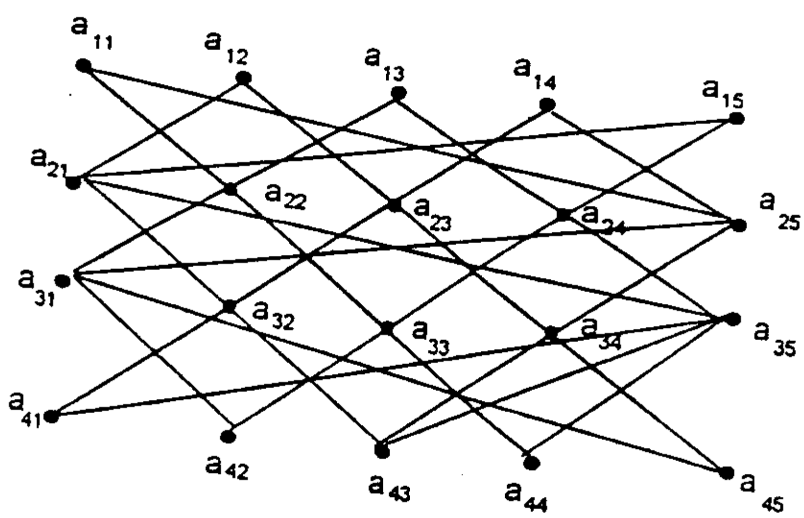
This can be seen in Figure 3a for $n = 12$. This can be really extended for any even n, $n \geq 4$.

Similarly, for n odd, we take $n = 13$ as an example , as shown in Figure 3b The result follows immediately from the well known fact that C_n is of type 1 if and only if $n \equiv 0 \bmod 3$.

(iii) We have to consider two cases

Case (1): $P_m \wedge C_{2r+1}$: This is a connected graph, see [7], which can be shown to be type 1 Figure 2a shows the colouring for $m = 4$ and $n = 5$.

$P_4 \wedge C_5$



$\cong$

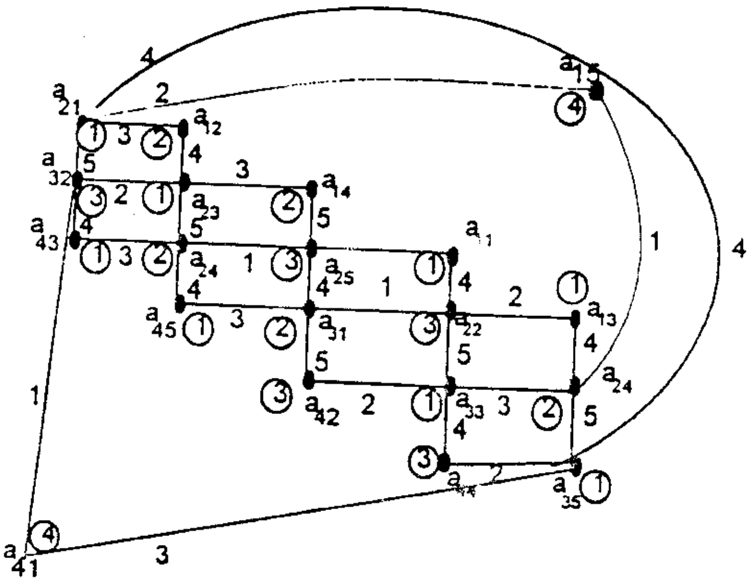
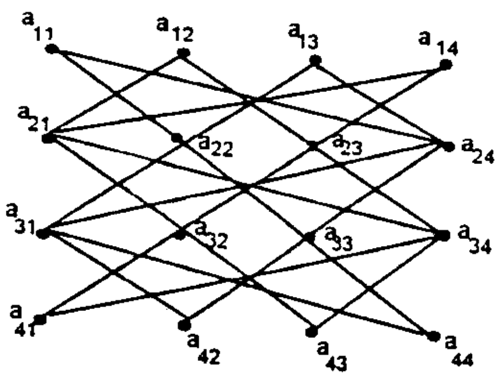


Figure 2a

Case (2): $P_n \wedge C_{2r}$: This is a disconnected graph, see [7], which can be shown to be of type 1. Figure 2b shows the colouring for m = 4 and n = 4.



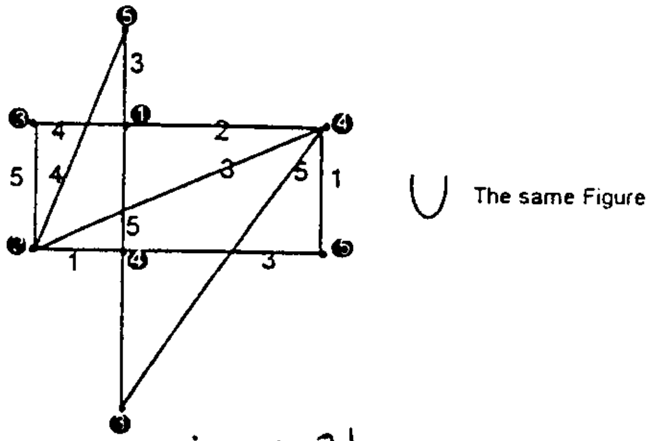
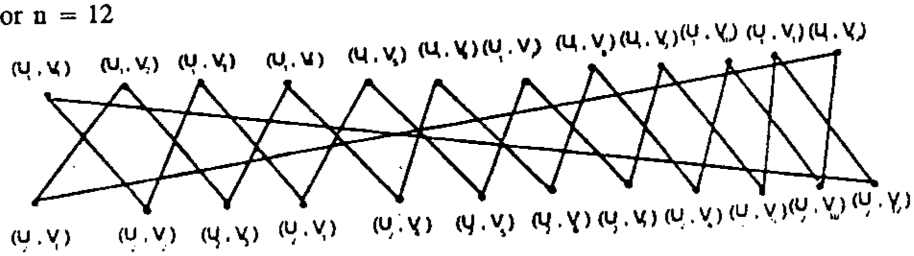


Figure 2b

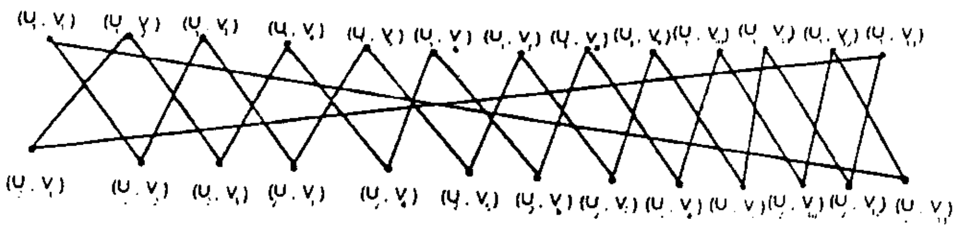
For n = 12



≅ 2 C_{12}

Figure 3a

For n = 13



$\cong C_{26}$

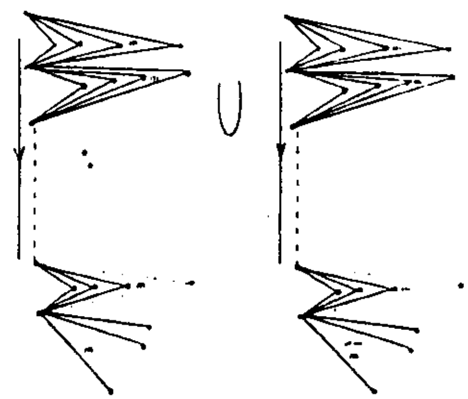
Figure 3b

Theorem 2 a

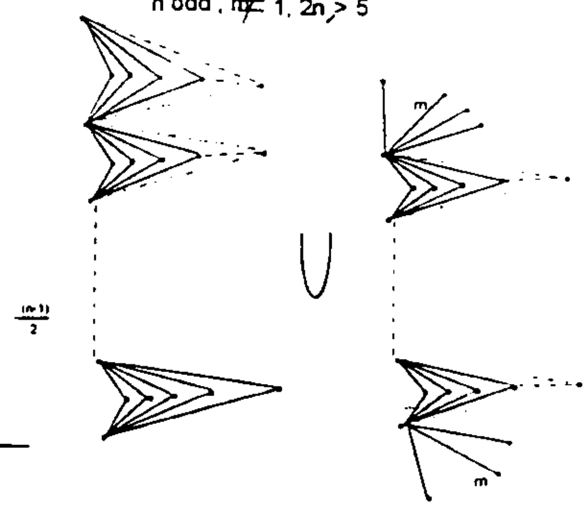
$P_n \wedge K_{1,m} =$

$2K_{1,m}$ $m \neq 1,2$, $n = 2$

$K_{1,2m} \cup K_{2,m}$ $m \geqslant 3$, $n = 3$



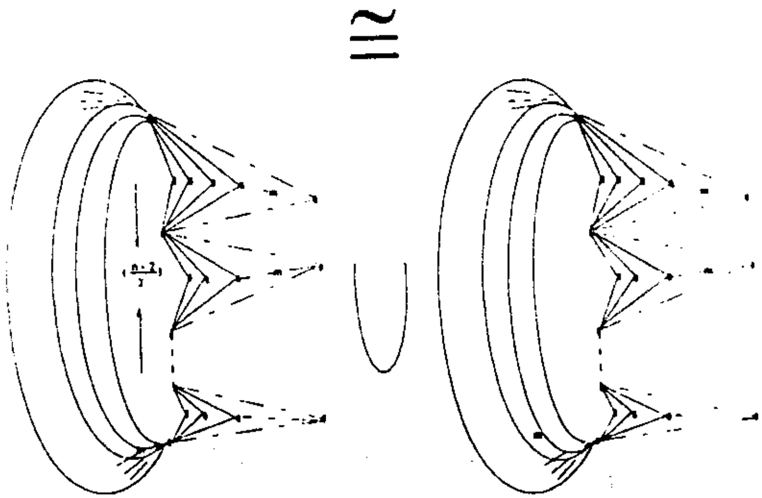
n odd , $m \neq 1$, $2n \geqslant 5$



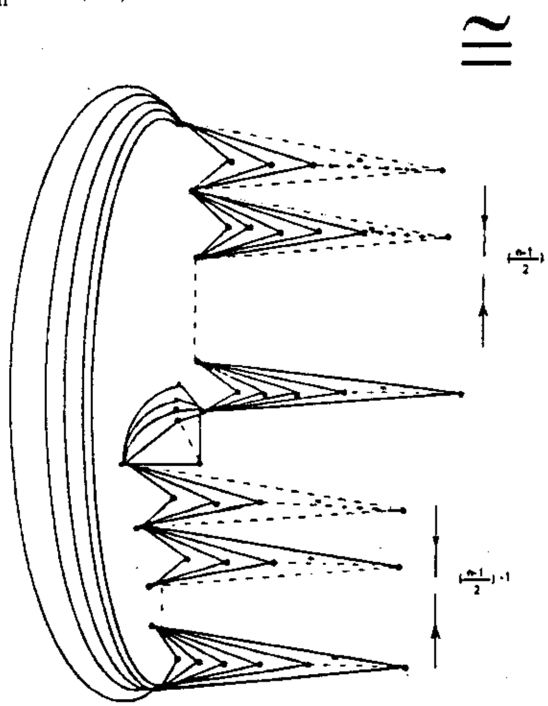
Proof: by induction on the number of vertices

Theorem 2 b

(i) $C_n \hat{} \, k_{1,m}$ where $m = 1,2$, n even



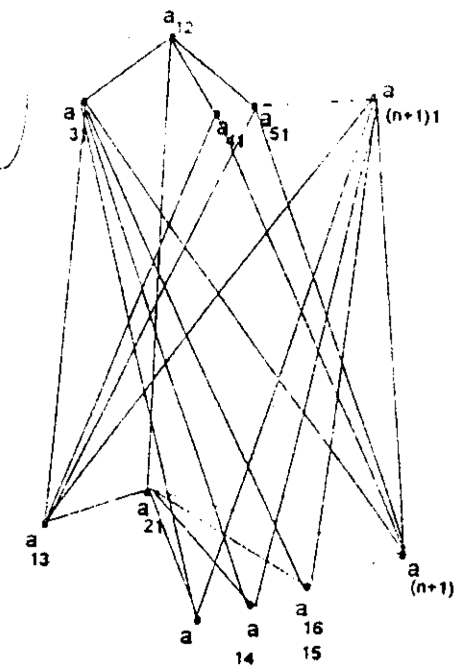
(ii) $C_n \hat{} \, K_{1,m}$ $m \mp 1,2$ n odd



Theorem 2 c

For n,m $\neq$ 1,2

$$K_{1,n} \backslash K_{1,m} \cong K_{1,nm} \cup$$



Proof: By induction on the number of vertices.

Theorem 3

All wheels W_n, $n \neq 3$, helms, webs and trees T_n, $n \neq 2$ are of type 1

Proof:

Figure 4 shows a method of colouring for wheels, helms and webs

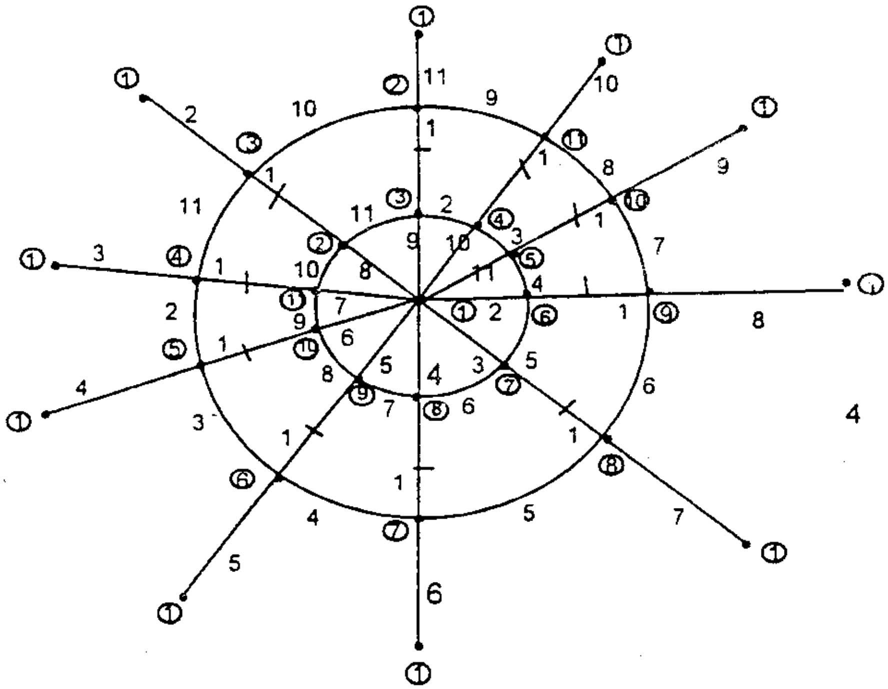
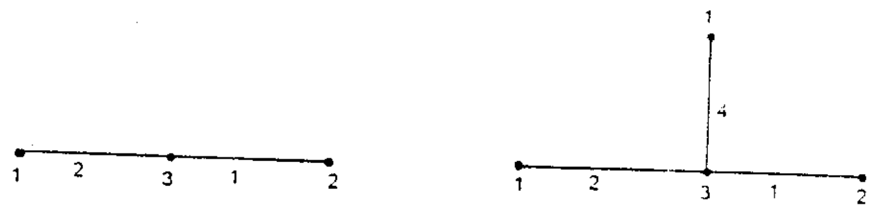


Figure 4

For trees at $n = 2$ then $T_2 = P_2$ which is of type 2 for $n \geq 3$ we prove the assertion by induction on the number of vertices of the tree

For n = 3 and n = 4 the following Figure shows that T_3 and T_4 are of type 1



Now let every tree T_k be of type 1, $k \geq 5$ Then any T_{r+1} may be obtained by adding a vertex u with joining it with a vertex v in the tree T_k and we have to consider the following two cases.

Case (i) $d(v) < \Delta T_k$ in the tree T_k, d(v) is the degree of the verte v in T_k

In This case, we label the added edge uv by the missing colour from those of the edges incident with v in T_k and we label the added verte u by a colour different from those of v or uv, this is available since d(v)= 1 in the tree T_{k+1}. Thus:

$$\chi''(T_{k+1}) = \chi''(T_k) = \Delta(T_k) + 1$$
$$= \Delta(T_{k+1}) + 1$$

and so T_{k+1} will be of type 1

Case (ii) $d(v) = \Delta(T_k)$ in the tree T_k

In this case, $\Delta(T_{k+1}) = \Delta(T_k) + 1$. We must label the added edge uv by $\Delta(T_k) + 2 = \Delta(T_{k+1}) + 1$, and we label the added vertex u by a suitable colour as in case (i). This

$$\chi''(T_{k+1}) = \chi''(T_k) + 1$$
$$= \Delta(T_k) + 2$$
$$= \Delta(T_{k+1}) + 1$$

and so T_{k+1} will be of type 1

So in all case Tn is of type 1, $n \neq 2$

Definition:

An elementary homomorphism of a graph G is an identification of two non adjacent vertices.

Lemma 1

Every circuit with one or two emerging paths from a vertex is of type 1

Proof:

For a circuit C_n, with $n = 0 \bmod 3$, the assertion is clear. Even for the case $n \neq 0 \bmod 3$, the assertion is also true.

Figure 5 shows the method of colouring

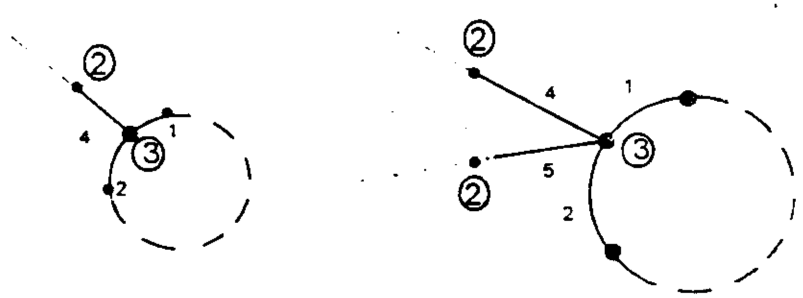


Figure 5

Remark:

The following graphs are of type 1:

i- Two circuit with a common vertex.

ii- Two sets each consisting of two multiple edges with a common vertex

iii- A circuit of a common vertex with two multiple edges

Theorem 4

(i) An elementry homomorphism on a path P_n induces a graph of type 2 if and only if the identification is between the end vertices of the path, and n = 0 mod 3 or n = 2 mod 3.

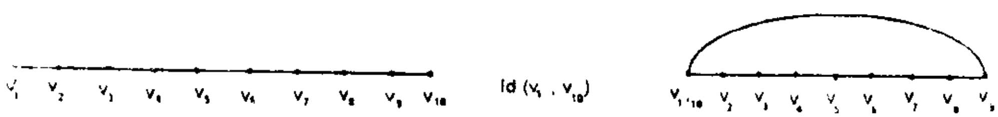
(ii) An elementry homomorphism on a tree $\neq P_n$ establishes a graph of type 1.

(iii) An elementry homomorphism on a circuit C_n establishes a graph of type 1.

Proof:

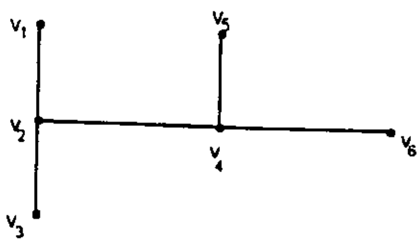
(i) Any identification of two vertices other than the end vertices gives rise to a circuit with one or two emerging path which is of type 1 according to Lemma (1). This identification constructs a circuit with n-1 vertices which is of type 2 if and only if $n-1 \equiv 1$ or 2 mod 3 which is equivalent to $n \equiv 2$ or 0 mod 3.

Example:



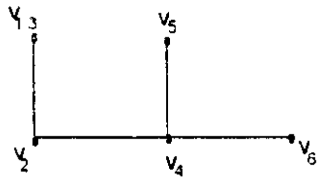
(ii) If the distance between the two vertices of identification is two, then we have a tree which is a graph of type 1. If the distance is greater than two, then we have a graph consisting of circuit with a tree (or more) emerging from one (or more) of its vertices, which can be proved, as in Lemma 1, to be of type 1.

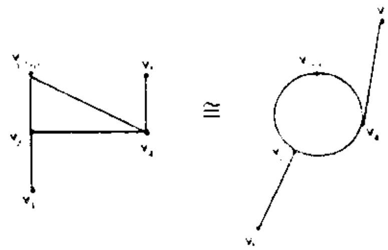
Example: T_6



$d(v_1, v_1) = 2$ $d(v_1, v_1) > 2$

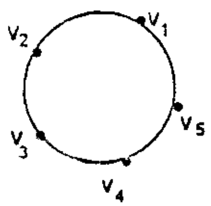
(i) $Id(v_1, V_3)$ (ii) $Id(v_1, v_6)$



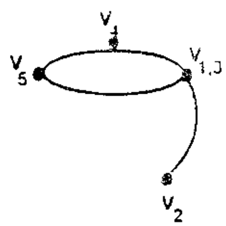


(iii) The resulting graph is one of the three graphs of the remark above which are all of type 1.

Example:



$Id(v_1, v_3)$



The square of cycle:

Theorem 5

The square of a cycle C_n is of type 1: , $4 \neq n \neq 7$

Proof:

For $n = 4$, $C^2{}_n$ is K_4, which is type 2. For $n = 7$, $C^2{}_n$ is a non-conformable graph, so it is type 2 [4]. Now

(1) For $n \equiv 0 \bmod 6$, Figure 6 a shows the method of colouring for $n = 12$.

(2) For n ≡ 1 mod 6, Figure 6b shows the method of colouring for n = 19

(3) For n ≡ mod 6, n ≡ 5 mod 6, Figure 6c, 6d show the method of colouring for n = 14,11.

(4) For n ≡ 3 mod 6, Figure 6e shows the method of colouring for n = 15.

(5) For n ≡ 4 mod 6, Figure 6h shows the method of colouring for n = 22.

C^{7}_{17}

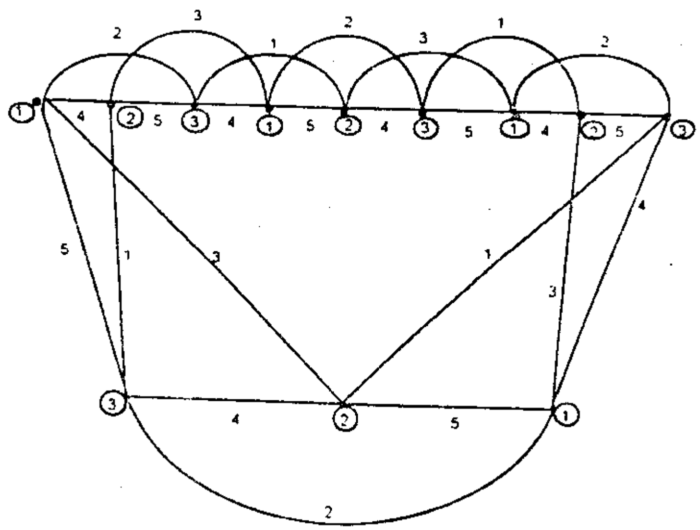


Figure 6 a

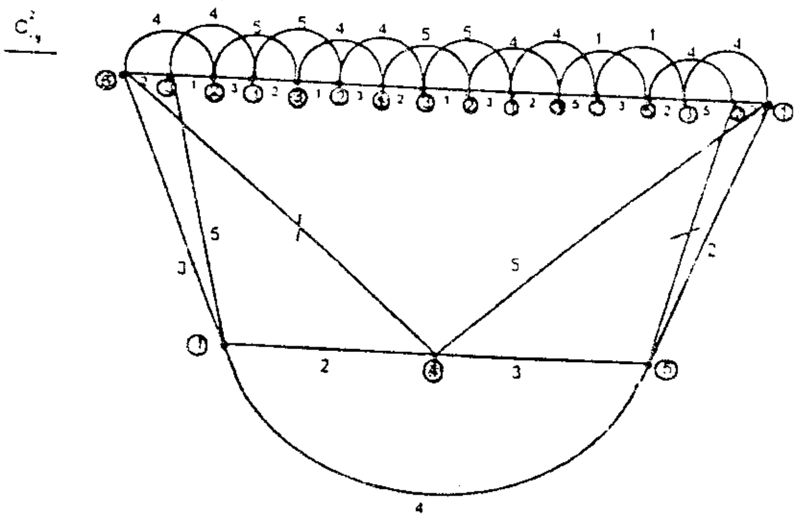


Figure 6 b

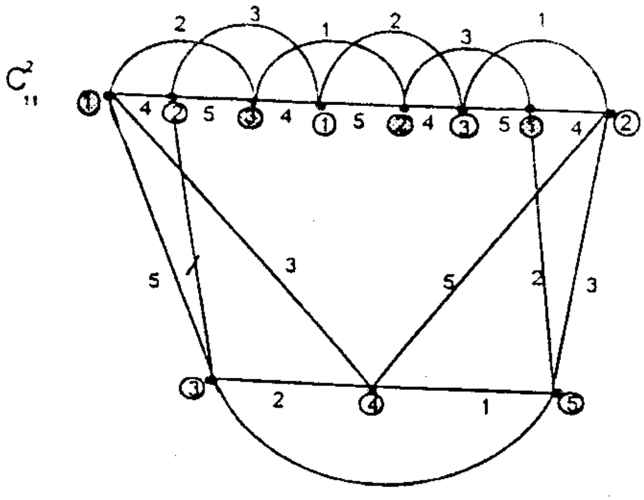


Figure 6 d

C^2_{15}

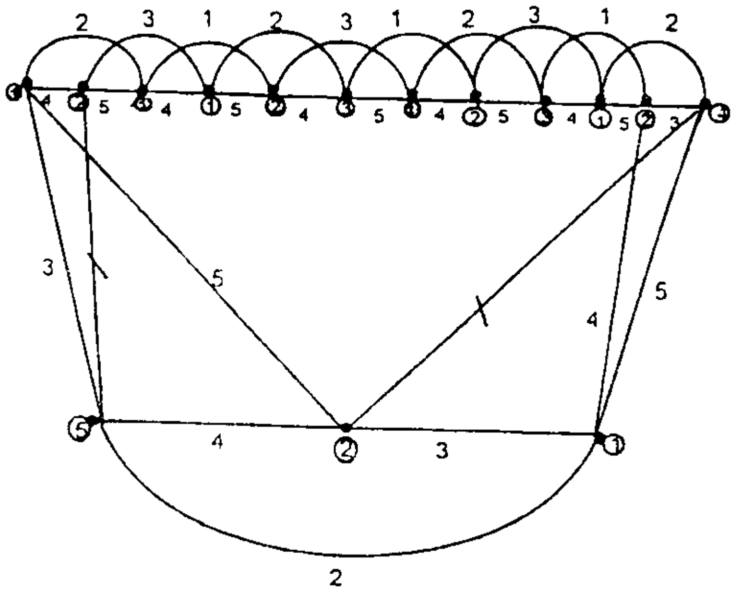


Figure 6e

C^2_{14}

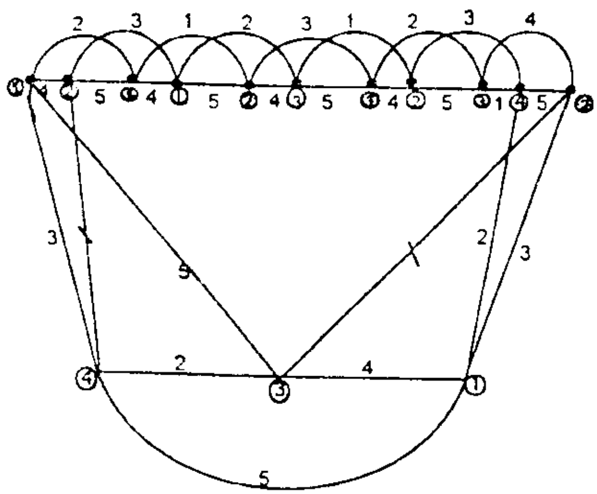


Figure 6c

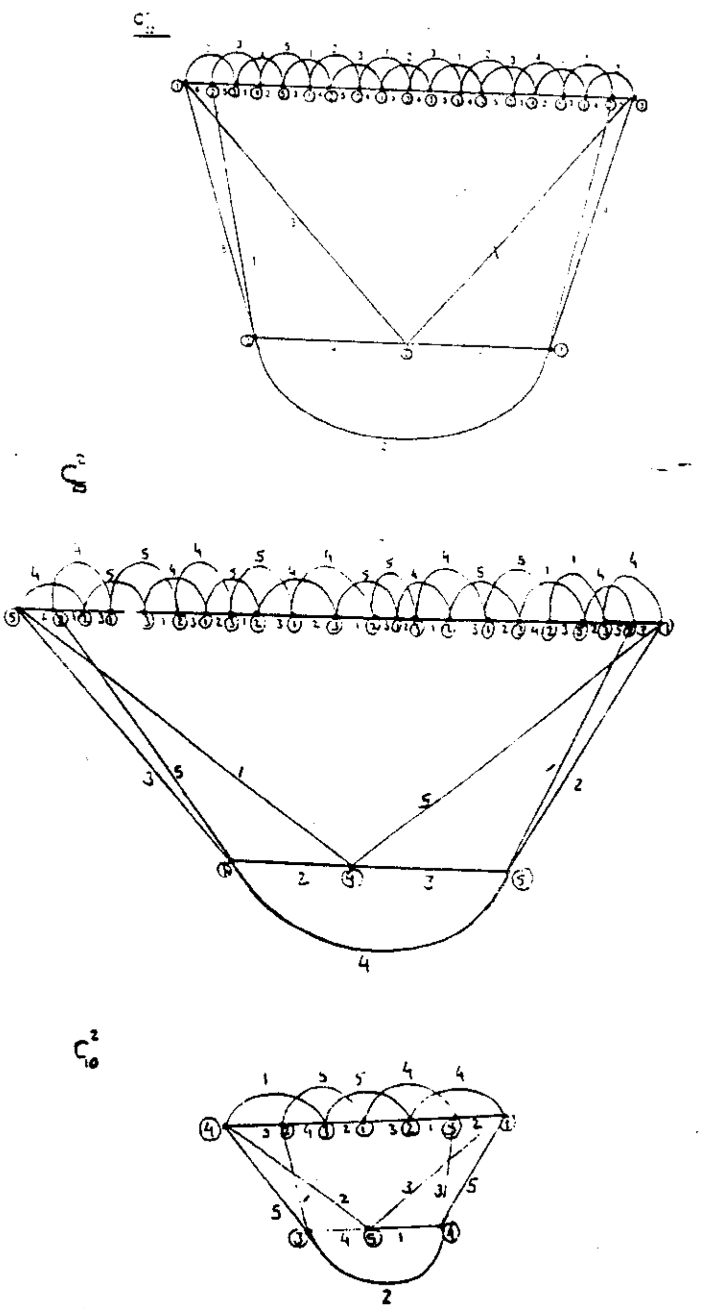


Figure 6b

Uniquely Totally Colourable Graphs

Let G be a totally labelled graph, i.e. its vertices and edges are distinguished from one another by names, such as $v_1, v_2, v_3, \ldots, v_n$ and $e_1, e_2, \ldots, e_m$ respectively. Any $\chi''(G)$ colouring of G induces a partition of the set of vertices and edges of G into $\chi''(G)$ colour classes. If $\chi''(G) = n$ and every colouring of G induces the same partition of the set of vertices and edges then we say that G is uniquely n - totally clourable or simply uniquely totally colourable.

Theorem 6

All paths and cycles Cn , n = O mod 3 are uniquely totally colourable

Proof:

It is immediate to notice that all paths are uniquely totally colourable. Now for circuits C_n , n = mod 3 , we show by induction "on the number of vertices "n"that they are uniquely totally colourable. The case n = 3 is trivial.

Let the assertion be true for n and we show its validity at n + 3. For this purpose make a cut at any vertex and insert a path of length 3 at the cut as in Figure 7 a.

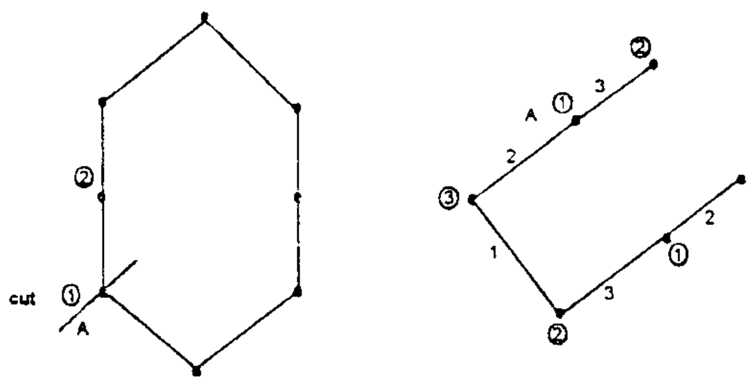


Figure 7 a

We consider now the case $C_n, n \neq o \bmod 3$, i.e. $n \equiv 1 \bmod 3$ or $n \equiv 2 \bmod 3$. First, We treat the case $n \equiv 1 \bmod 3$, where n is even.

The example C_{10} as ahown in Figure 7 b indicates two methods of colouring which are also relevant for similar situation.

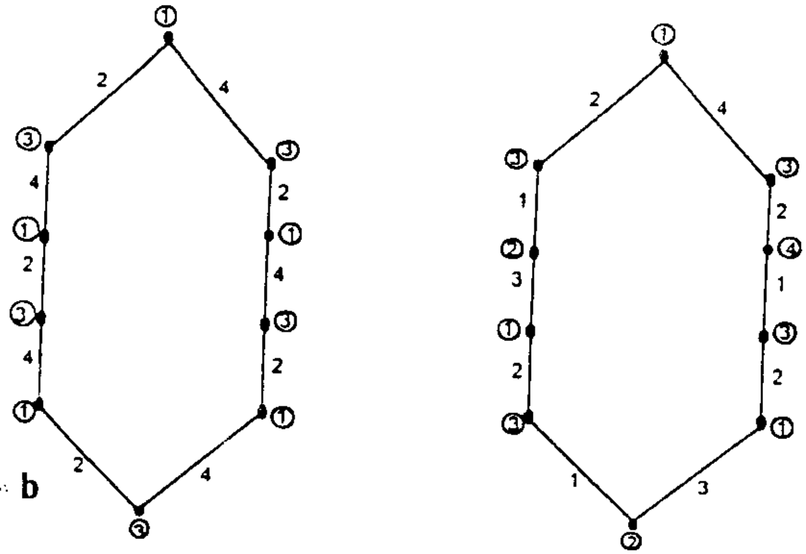


Figure 7 b

For n odd, example C_7 illustrates two methods of colouring.

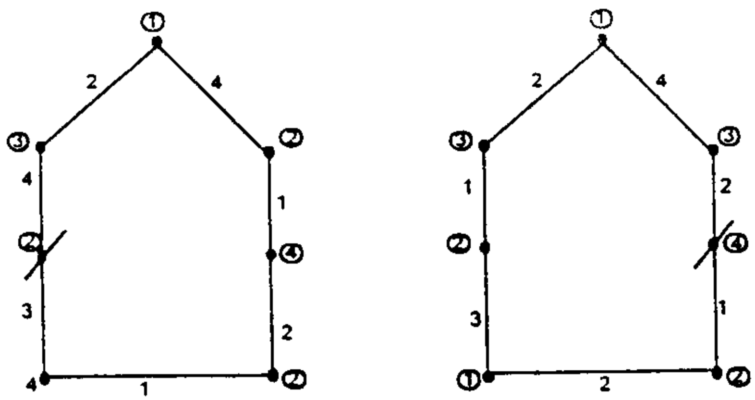


Figure 7 c

Second, we study the case $n \doteq 2 \bmod 3$, where n is even. We take C_8 to illustrate two methods of colouring :

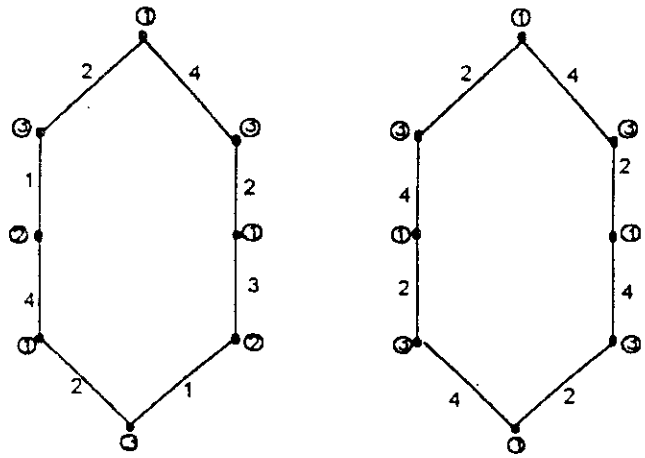


Figure 7 d

Now for n odd, we take C_{11} as an example to show two methods of coloruing.

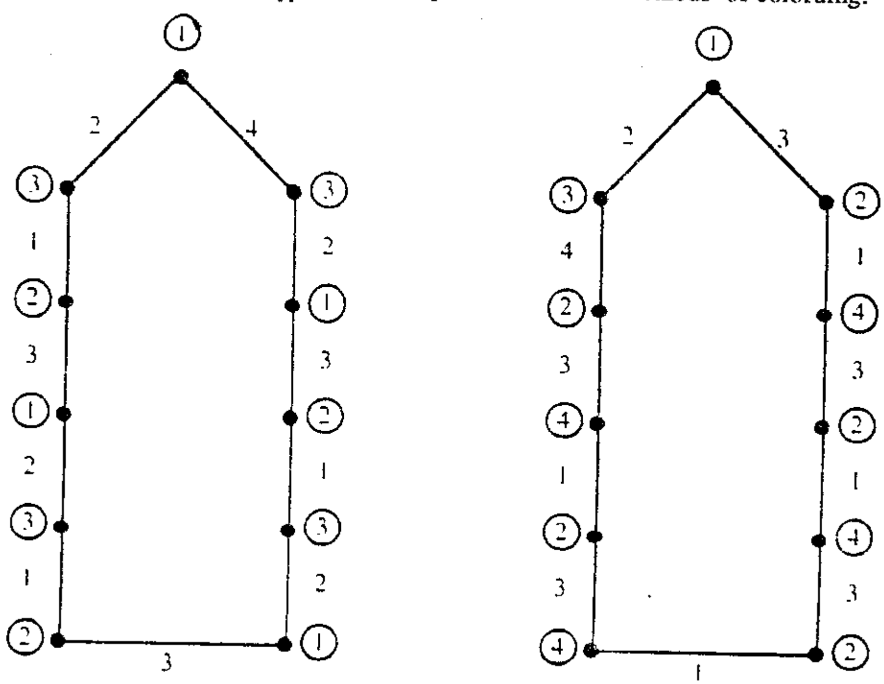


Figure 7e

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بعض الملاحظات على رقم التلوين الكلى

السيد أنور السعيد محمد السخاوى
قسم الرياضيات - كلية العلوم - جامعة عين شمس

فى هذا البحث تم تعين رقم التلوين الكلى لاتصال مسارين ولاتصال مسار P_m مع دوارة C_n. كما تم ايجاد رقم التلوين الكلى لكل من $K_{1,m}$ ^ C_n حيث $n, m \neq 1,2$ عدد زوجى، $K_{1,m} \wedge K_{1,n}$ وأثبت أن رقم التلوين الكلى للعجلات W_n (حيث $n \neq 3$) وللأشجار T_n (حيث $n \neq 2$)هو $n+1$.

ولقد تم دراسة لتأثير الهوميومورفزم الأولى على رقم التلوين للمسارات والأشجار والدوارات . وتم تعريف مفهوم التلوين الكلى الوحيد للرسوم ومن هذا التعريف استنتج مباشرة أن كل المسارات والدوائر C_n ,$n \equiv 0 \bmod 3$ ذات تلوين كلى وحيد.