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REGULARITY AND NORMALITY IN FUZZY TOPOLOGICAL ORDERED SPACES

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ABSTRACT

This paper provides a hierarchy of some separation axioms for the fuzzy topological ordered spaces

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1- INTRODUCTION

In [3] the second outher constructed the concepts of fuzzy hausdorff topological ordered spaces and some of the weakest fuzzy separation axioms of it such as FT_0 - and FT_1 - ordered spaces.

In this paper we will define and study some separation axioms for the fuzzy topological ordered spaces. Section 2 contains some preliminaries. Section 3 is devoted to define and study the notion of fuzzy regular and FT_3 - ordered spaces. Section 4 devoted to define and study the notion of fuzzy completely regular ordered spaces which reduce to the corresponding ones of Hutton [5] when relation is discrete order (i.e., $x \leq y$ iff $x = y$). In section 5 we will more study of the concept of Katsaras's normality [7]. In section 6 we will define and study the complete normality of the fuzzy topological ordered space and $FT_{11/2}$ -ordered space. Finally, it is worth mentioning that some separation axioms in the sence of fuzzy topology are contained as particular cases in the corresponding ones in the sence of order and fuzzy toological structures.

2- PRELIMINARIES

A fuzzy set in X is called X a fuzzy point [9] iff it takes the value o for all $y \in X$ except one, say $x \in X$, is t(o $< t \leq 1$), we denote this fuzzy point by μ_x^t , where the point x is called its support. Also, a fuzzy point μ_x^t is said to be contained in a fuzzy set λ or to be belong to λ, denoted by $\mu_x^t \leq \lambda$, iff $t \leq \lambda(x)$. Evidently every fuzzy set can be be expressed as the union of all fuzzy points which belong to it.

A fuzzy topology [2] on X is a subfamily τ of I^X such that:

(i) $o, 1 \in \tau$,

(ii) If $\mu, \rho \in \tau$, then $\mu \wedge \rho \in \tau$,

(iii) If $\mu_\alpha \in \tau$ for each $\alpha \in \rho$, then $\sup_{\alpha \in J} \{\mu_\alpha\} \in \tau$.

A fuzzy topological preordered (resp. ordered) space, which given sympolically as $(X, \leq, \tau)$, is defined to be a non-empty set X endwed with preorder (resp.order) relation $\leq$ and a fuzzy topology τ both given on X. Since the category of preordered sets constitutes a more generally than the category of ordered (i.e., anti-symmetric preordered) sets, so that we habitually treat with the category of preordered sets which reduces, automatically, to the category of ordered sets when the anti-symmetric condition holds.

A function f of a fuzzy topological preordered space $(X, \leq, \tau)$ to a fuzzy topological preordered space $(Y, \leq, \nu)$ is said to an order-preserving fuzzy continuous function if the inverse image of each increasing (resp.decreasing) ν-open fuzzy set is increasing (resp. decreasing) τ-open fuzzy set.

For an arbitrary fuzzy set λ in a preordered set $(X,\leq)$ the smallest increasing (resp. decreasing) fuzzy set containing λ is called the increasing (resp.decreasing) hull and denoted by $i(\lambda)\{resp.d(\lambda)\}$ (cf. Katsaras [7]). Also there exists a smallest closed increasing (resp. decreasing) fuzzy set containing λ in $(x,\leq)$ such closed fuzzy set is called the closed increasing (resp.decreasing) hull and denoted as $I(\lambda)\{$ resp.$D(\lambda)$ $\}$, whenever:

$I(\lambda) = \inf\{\nu : \nu \geq \lambda, \nu$ is closed and increasing fuzzy set$\}$

$D(\lambda) = \inf\{\mu : \mu \geq ; \mu$ is closed and decreasing fuzzy set$\}$

Definition 2.1 [3] A fuzzy topological preordered space $(X, \leq, \tau)$ is said to be sFT_1 - preordered space if for each fuzzy point μ_x^r in the preordered set $(X, \leq)$ both $i(\mu_x^r)$ and $d(\mu_x^r)$ are τ-closed fuzzy sets.

Definition 2.2 [3] A fuzzy topological preordered space $(X, \leq, \tau)$ is said to be fuzzy Hausdorff fuzzy topological (FT_2-) preordered space iff, for each pair of fuzzy points $\mu_x^r, \mu_y^t \in I^X$ with $x \leq y$ (resp. $x \geq y$) in a preordered set $(x, \leq)$, there exist an increasing (resp.decreasing) τ-open fuzzy set μ and a decreasing (resp.increasing) τ-open fuzzy set λ such that $r \leq \mu(x)$, $t \leq \lambda(y)$ and $\mu \leq (1-\lambda)$.

Definition 2.3 [1] Let $(X,\leq,\tau)$ be a fuzzy topological preordered space. A subset A of the preordered set $(X,\leq)$ is said to be preordered fuzzy subspace of $(X,\leq,\tau)$ if A gives both the induced preorder $\leq_\Delta$ and the relative fuzzy topology $\tau_\Delta = \{\lambda/A : \lambda \in \tau\}$ and we denote it by $(A, \leq_\Delta, \tau_\Delta)$.

Definition 2.4 [1] Let $\{(x_\alpha, \leq_\alpha, \tau_\alpha): \alpha \in Q\}$ be an indexed family of fuzzy topological preordered spaces, let $x = \prod_{\alpha \in J} x_\alpha$ be the usual cartesian product of x_α, s and $\leq$ be the preorder relation given as $x \leq y$ iff, $P_\alpha(x) \leq_\alpha p_\alpha(y)$ for each $\alpha \in J$ where
$P_\alpha : X = \prod_{\alpha \in J} X_\alpha \dashrightarrow X_\alpha$ is the usual projections of the cartesian product space $X = \prod_{\alpha \in \Pi} X_\alpha$ to the α-coordinate space x_α. The smallest fuzzy topology on $x = \prod_{\alpha \in J} x_\alpha$ which makes each projection p_α to be fuzzy continuous is the fuzzy topology $\tau = \prod_{\alpha \in J} \tau_\alpha$ and the triple $(x, \leq, \tau)$ is called the product fuzzy topological preordered space.

It is clear that the projections $p_\alpha : X = \Pi X_\alpha \dashrightarrow X_\alpha$ are order - preserving fuzzy contiuous functions.

Next, we recall the definition of the fuzzy unit interval which was given in [4]. Let J_Φ denote the set of all decreasing functions

$$f: [0,1] \text{--------}> [0,1]$$

On J_Φ we consider the equivalence relation $\sim$ defined by

$f_1 \sim f_2$ iff $f_1(t+) = f_2(t+)$ and $f_1(t-) = f_2(t-)$ for all $t \in [0,1]$.

The fuzzy unit interval is the set $[0,1]_\Phi$ of all equivalence classes in which J_Φ is partioned by $\sim$. For each $t \in (0,1)$ we define L_t and R_t on $[o,1]_\Phi$ by: $L_t(f) = 1-f(t-)$ and $R_t(f) = f(t+)$.

The fuzzy topology on $[o,1]_\Phi$ which is generated by the family $\{L_t, R_t : t \in (o,1)\}$ is called the usual fuzzy topology on $[o,1]_\Phi$.

By the usual order on $[o,1]_\Phi$ we will mean the partial order $\leq$ defined by:

$f_1 \leq f_2$ iff $f_1(t+) \leq f_2(t+)$ and $f_1(t-) \leq f_2(t-)$ for $t \in [o,1]$.

3- Fuzzy regular and FT_3- ordered spaces

Definition 3.1 A fuzzy topological preordered space $(x, \leq, \tau)$ is said to be fuzzy regular preordered iff, the following condition holds: Given a fuzzy point μ_x^r in a preordered set $(x, \leq)$ and a decreasing (resp. increasing) τ-open fuzzy set λ containing μ_x^r, there exist a decreasing (resp. increasing) τ-open fuzzy set μ and a decreasing (resp. increasing) τ-closed fuzzy set ν such that

$$\mu_x^r \leq \mu \leq \nu \leq \lambda$$

If the preorder of the fuzzy regular preordered space $(x, \leq, \tau)$ is an order, then $(x, \leq, \tau)$ is called a fuzzy regular ordered space.

Proposition 3.1 A fuzzy topological preordered space $(x, \leq, \tau)$ is said to be fuzzy regular preordered iff, the following condition holds: Given a fuzzy point μ_x^r in a preordered set $(x, \leq)$ and a decreasing (resp. increasing) τ-open fuzzyset λ containing μ_x^r there exists a decreasing (resp. increasing) τ-open fuzzy set μ such that

$$\mu_x^r \leq \mu \leq D(\mu) \leq \lambda \{ \text{ resp. } \mu_x^r \leq \mu \leq I(\mu) \leq \lambda \}$$

Proof : Striaghtforward.

Proposition 3.2 Any preordered fuzzy subspace of a fuzzy regular preordered space is a fuzzy regular preordered space.

Proof : Let $(A, \leq_\Delta, \tau_\Delta)$ be a preordered fuzzy subspace of fuzzy regular preordered space $(x, \leq, \tau)$ and let μ_x^r be a fuzzy point, in $(A, \leq_\Delta, \tau_\Delta)$, contained in a decreasing τ_Δ -open fuzzy set μ. So that there exists a decreasing τ-open fuzzy set λ containing μ_x^r, in the preordered set $(x, \leq)$, with $\mu = \lambda/A$. Regularity of $(x, \leq, \tau)$ implies that there exists a decreasing τ-open fuzzy set ν such that

$$\mu_x^r \leq \nu \leq D(\nu) \leq \lambda$$

Restriction with respect to A implies that

$$\mu_x^r \leq ([\nu/A) \leq (D(\nu)/A) \leq (\lambda/A)$$

If $(\nu/A) = \nu^*$ and $(D(\nu)/A) = D_\Delta(\nu^*)$. Then for a decreasing τ-open fuzzy set μ containing the fuzzy point μ_x^r, there is a decreasing τ_Δ -open fuzzy set ν^* such that $\mu_x^r \leq \nu^* \leq D(\nu^*) \leq \mu$. Then $(A, \leq_\Delta, \tau_\Delta)$ is a fuzzy regular preordered space.

Preposition 3.3 The product of a family of fuzzy regular preordered spaces $\{(x_\alpha, \leq_\alpha, \tau_\alpha): \alpha \in J\}$ is again fuzzy regular preordered space.

Proof: Let $\{(x_\alpha, \leq_\alpha, \tau_\alpha): \alpha \in J\}$ be an indexed family of fuzzy regular preordered spaces, let μ_x^r be a fuzzy point in the product fuzzy topological preordered space $(x, \leq, \tau)$ contained in a decreasing τ-open fuzzy set λ. Since the projections $p_\alpha : x \text{-------}> x_\alpha$ are order-preserving fuzzy continuous functions, then it guarantees that the fuzzy points μ_α contained in decreasing τ_α -open fuzzy sets λ_α, for each $\alpha \in J$, in $\{(x_\alpha, \leq_\alpha, \tau_\alpha): \alpha \in J\}$ such that $\lambda = \{p_\alpha^{-1}(\lambda_\alpha): \alpha \in J\}$ regularity of $\{(x_\alpha, \leq_\alpha, \tau_\alpha): \alpha \in J\}$ implies that there exist a family of decreasing τ_α-open fuzzy sets ν_α such that.

$$\mu_{x_\alpha} \leq_\alpha \nu_\alpha \leq D(\nu_\alpha) \leq \lambda_\alpha$$

and it follows that for a fuzzy point p and a decreasing τ-open fuzzy set $\lambda = \{p_\alpha(\lambda_\alpha): \alpha \in J\}$ containing the fuzzy point μ^r_x, there exist decreasing τ-open fuzzy set $\{\Omega_\alpha(\nu_\alpha): \alpha \in J\}$ and a decreasing τ-closed fuzzy set $\{p_\alpha(D(\nu_\alpha)): \alpha \in J\}$ such that

$$\mu^r_{x_\alpha} \leq p_\alpha^{-1}(\nu_\alpha) \leq p_\alpha^{-1}(D(\nu_\alpha)) \leq \lambda$$

Then the product fuzzy topological preordered space $(X, \leq, \tau)$ is a fuzzy regular preordered space.

Definition 3.2 A fuzzy regular preordered space wich is also SFT_1-preordered space is called a FT_3 -preordered space.

proposition 3.4 A FT_3 -preordered space is a FT_2 -preordered space.

proof: Straightforward.

4.Fuzzy completely regular and $Ft_{7/2}$-ordered spaces.

Definition 4.1 A fuzzy topological preordered space $(X, \leq, \tau)$ is said to be fuzzy completely regular preordered space if for each decreasing (resp.increasing) τ-open fuzzy set λ in $(X, \leq, \tau)$,there exist a collection λ_i of fuzzy sets and order-preserving fuzzy continuous functions $f_i: X \text{----}> [0,1]$, such that,for each $x \in X$, $\lambda = \sup\{\lambda_i\}$; and

$$\lambda_i(x) \leq 1-f(x)(o+) \leq 1-f(x)(1-) \leq \lambda(x).$$

If the preorder of the fuzzy regular preordered space $(x, \leq, \tau)$ is an order then $(x, \leq, \tau)$ is called a fuzzy regular ordered space, also if such preorder is the discrete order, then the fuzzy regular preordered space reduces to the Hutton's complete regularity in the senc fuzzy topology [5].

Proposition 4.1 Any preordered fuzzy subspace of a fuzzy completely regular preordered space is again fuzzy completely regular preordered space.

Proof: Let μ be a decreasing τ_Δ -open fuzzy set in $(A, \leq_\Delta, \tau\Delta)$, then there is a decreasing τopen fuzzy set λ in the fuzzy completely regular preordered space $(X, \leq, \tau)$ such that $\mu = \lambda/A$. Complete regularity of $(X, \leq, Y\tau)$implies that there are a collection λ_i of fuzzy sets and order-preserving fuzzy continuous functions

$$f_i : X \text{-----}> [o,1]_*$$

such that ,for each $x \epsilon X, \lambda = \sup\{\lambda_i\}$ and

$$\mu_i(x) \leq 1\text{-}f(x)(o+) \leq 1\text{-}f(x)(1\text{-}) \leq \lambda(x).$$

Hence one can notice that there exist both the collection $\nu_i = (\lambda_i)/A$ and the order-preserving fuzzy continuous restrictions

$$(f_i)/A \ : X \text{----}> [o.1]_*$$

such that, for $y \in Y$, $\nu = \sup\{\nu_i\}$ and

$$\nu_i(y) \leq i - \{(f_i)/ A\} (y)(o+) \leq 1 - \{(f_i)/A\}(y)(1\text{-}) \leq \nu(y).$$

Then $(A, \leq_\Delta, \tau_\Delta)$ is a fuzzy completey regular preordered space.

Proposition 4.2 Let $\{(X_\alpha , \leq_\alpha , \tau_\alpha) : \alpha \in J\}$ be an indexed family of fuzzy completely regular preordered spaces. Then the product fuzzy topological reordered space $(\Pi_{\alpha \epsilon J} X_\alpha , \leq , \tau)$is a fuzzy completely regular preordered space.

Proof: Let $\{(x_\alpha, \leq_\alpha, \tau_\alpha): \alpha \in J\}$ be afamily of fuzzy completely regular preordered spaces, for each decreasing τ_α -open fuzzy sets λ_α, there exist a collection $\{(\lambda_\alpha)_i : \alpha \in J\}$ of fuzzy sets and order-preserving fuzzy contiuous functions

$$(f_\alpha)_i : x_\alpha \text{-------}> [o,1]_*$$

such that, for $x \in X$, $\lambda_\alpha = \sup (\lambda_\alpha)_i$ and

$$(\lambda_\alpha)_i(x) \leq 1\text{-}(f_\alpha)_i(x)(o+) \leq 1\text{-}(f_\alpha)_i(x)(1\text{-}) \leq \lambda_\alpha(x).$$

Since each projection of $P_\alpha : X \text{------}> X_\alpha$ is an order-preserving fuzzy continuous function, then one can say that for a decreasing τ-open fuzzy set $\lambda = \{P_\alpha (\lambda_\alpha): \alpha \epsilon J\}$, there exist a collection $\lambda_i = \{p_\alpha^{-1} (\lambda_\alpha): \alpha \epsilon J\}$; with $\lambda = \sup(\lambda_\alpha)$, and order-preserving fuzzy continous compositions $((f_\alpha)_i o P_\alpha)$: X ---- $[o,1]_\Phi$ with

$$p_\alpha^{-1}\{(\lambda_\alpha)_i(x) \leq 1-(f_\alpha)_i (x)(o+) \leq 1-(f_\alpha)_i(x)(1-) \leq \lambda_\alpha(x)$$

which implies that

$$\lambda_i(x) \leq 1-((f_\alpha)_i \text{ o } p_\alpha)(x)(o+) \leq 1-((f_\alpha)_i \text{ o } p_\alpha)(x)(1-) \leq \lambda(x)$$

and this completes the proof.

Proposition 4.3 A fuzzy completely regular preordered space is a fuzzy regular preordered space.

Proof: Let $(x, \leq, \tau)$ be a fuzzy completely regular preordered space and let μ be a decreasing τ-open fuzzy set. Then there exist a collection λ_i of fuzzy sets and order-preserving fuzzy continuous functions f_i : x ----> $[o,1]_\Phi$ such that, for $x \epsilon X$,
$\lambda = \sup \lambda_i$ and

$$\lambda_i(x) \leq 1-f_i(x)(o+) \leq 1-f_i(x)(1-) \leq \lambda(x)$$

So that for an arbitrary fuzzy point μ_x^r in the fuzzy set λ one can found at least an order-preserving fuzzy continuous function, say

$$f: X \text{-------}> [o,1]_\Phi$$

such that, for $x \epsilon X$,

$$\mu_x^r(x) \leq 1-f(x)(o+) \leq 1-f(x)(1-) \leq \lambda(x).$$

Since f(x) is decreasing, then

$$f(x)(1-) \leq f(x)(1+/2) \leq f(x)(1-/2) \leq f(x)(o+)$$

Then $1-f(x)(1-/2) = 1-f^{-1}(R_{1/2}) = f^{-1}(1-R_{1/2})$ and it follows that

$$\mu_x^r \leq f^{-1}(L_{1/2}) \leq f^{-1}(1-R_{1/2}) \leq \lambda$$

Since $L_{1/2}$ and $(1-R_{1/2})$ are, respectively, decreasing open and closed fuzzy sets in

$[o,1]_*$ and the function f is an order-preserving fuzzy continuous, then $f^{-1}(L_{1/2})$ and $f^{-1}(1-R_{1/2})$ are, respectively, decreasing τ-open and τ-closed fuzzy sets in $(X,\leq,\tau)$. Then $(x,\leq,\tau)$ is a fuzzy regular preordered space.

Definition 4.2 A fuzzy completely regular preordered space which is also SFT_1 -preordered space is called $FT_{7/2}$-preordered space.

We omite the proof of the following easily established.

Lemma 4.1 A $FT_{7/2}$ -preordered space is FT_3 -preordered space.

5- Fuzzy normally ordered and FT_4- ordered spaces

Definition 5.1 [7] A fuzzy topological preordered space $(x,\leq,\tau)$ is said to be fuzzy normally preordered iff, the following condition holds: Given a decreasing (resp. increasing) τ-closed fuzzy set μ and a decreasing (resp. increasing) τ-open fuzzy set λ with $\mu\leq\lambda$, there exist a decreasing (resp. increasing) τ-closed fuzzy set γ such that

$$\mu \leq \nu \leq \gamma \leq \lambda$$

If the preordered of the fuzzy normally preordered space $(x,\leq,\tau)$ is an order, then $(x,\leq,\tau)$ is called fuzzy normally ordered space.

Proposition 5.1 Each closed preordered fuzzy subspace of a fuzzy normally preordered space is a fuzzy normally preordered space.

Proof : Let λ_1 and λ_2 be, respectively, a decreasing τ_Δ-closed and τ_Δ-open fuzzy sets with $\lambda_1\leq\lambda_2$ in the closed preordered fuzzy subspace $(A,\leq_\Delta,\tau_\Delta)$ of the fuzzy normally preordered space $(x,\leq,\tau)$. Since λ_1 and $(1-\lambda_2)$ are τ_Δ -closed fuzzy sets, then λ_1 and

λ_2 are, respectively, decreasing τ-closed and τ-open fuzzy sets with $\lambda_1 \leq \lambda_2$ in $(x, \leq, \tau)$. Fuzzy normality of $(x, \leq, \tau)$ implies that there exist a decreasing τ-open fuzzy set ν and a decreasing τ-closed fuzzy set γ such that $\lambda_1 \leq \nu \leq \gamma \leq \lambda_2$ and it follows that $\nu_1 = ((1-\nu)/A$ is an increasing τ_Δ -closed fuzzy set and $\gamma = (\Delta/A)$ is a decreasing τ_Δ-closed fuzzy set with $(1-\nu_1) \leq \gamma_1$ and $\lambda_1 \leq (1-\nu_1) \leq \gamma_1 \leq \lambda_2$. Then $(A, \leq_\Delta, \tau_\Delta)$ is afuzzy normally preordered space.

Proposition 5.2 Fuzzy normality of a fuzzy topological preordered space is not productive.

Proof: To prove this proposition we gave the following:

Counter example. Let $X = [o,1]$ and define fuzzy sets on X as.

$\mu = o$

$$\nu_1 = \begin{cases} o & o \leq x \leq 5.5 \\ \{(x-11/20)/2\} & 5.5 < x \leq 1 \end{cases}$$

$$\mu_1 = \begin{cases} x & o \leq x \leq o.5 \\ 9/2 & o.5 < x \leq 1 \end{cases}$$

$\nu = 1$

One can define a fuzzy topology, $\tau_1 = \{o, (1-\mu), \nu_1, 1\}$, on $X = [o,1]$.

Also we define

$\mu^* = o$

$$\nu_1^* = \begin{cases} o & o \leq x < o.5 \\ (2x-1) & o.5 \leq x \leq 1 \end{cases}$$

$$\mu_1^* = \begin{cases} \lambda & 0 \le x \le 0.5 \\ & 0.5 \le x \le 1 \end{cases}$$

$$\nu_1^* = 1$$

and we gave $\tau_2 = \{o, (1-\mu_1^*), \nu_1^*, (1-\mu^*) \vee \nu^*, 1\}$ to be a fuzzy topology on X. With the natural order on $X=[o,1]$ one can easily show that $(X, \le, \tau_1)$ and $(X, \le, \tau_2)$ are fuzzy normally preordered spaces because for an increasing τ_1-closed fuzzy set $\mu = o$ and an increasing τ_1-open fuzzy set $\nu = 1$ there exist an increasing τ_1-open fuzzy set ν_1 and an increasing τ_1-closed fuzzy set μ_1 such that $\mu \le \nu_1 \le \mu_1 \le \nu$.

But for a product fuzzy topological preordered space $X \times X, \le \tau_1 x \tau_2)$ with the fuzzy topology $\tau_1 x \tau_2 = \{o, \nu_1 x \nu_1^*, (1-\mu), (\nu_1 x \nu_1^*) \wedge (1-\mu_1), 1\}$. one can notice that for an increasing $\tau_1 x \tau_2$ -closed fuzzy set, say $\lambda = o$ and an increasing $\tau_1 x \tau_2$-open fuzzy set λ_1 and an increasing $\tau_1 x \tau_2$ -closed fuzzy set ν_1 such that $\lambda \le \lambda_1 \le \nu_1 \le \nu$. So we can say that the fuzzy normality of fuzzy topological preordered spaces is not productive.

Definition 5.2 A fuzzy normally preordered space which is also SFT_1 -preordered space is called FT_4 -preordered space.

Proposition 5.3 Every FT_4-preordered space is a fuzzy completely regular preordered space.

Proof: Let $(X, \le, \tau)$ be a FT_4 -preordered space and μ be a decreasing τ-open fuzzy set which is the union of each of its fuzzy points. Let μ_x^r be a fuzzy point in μ. Since $(X, \le, \tau)$ is SFT_1 -preordered space, then $i((\mu^r{}_{x_1}))$ is τ-closed fuzzy sets with

$(\mu'_{x_1}) \leq d((\mu'_{x_1})) \leq \mu$. Fuzzy normality of $(X, \leq, \tau)$ implies that for a decreasing τ-closed fuzzy set $d((\mu'_{x_1}))$ and a decreasing τ-open fuzzy set μ with $d(\mu_X) \leq \mu$, there exists, in view of [7, theorem 5.3] an order-preserving fuzzy continuous function $f_1 : X \longrightarrow [o,1]_s$ such that.

$$d((\mu'_{x_1}))(\chi) \leq 1-f_i(\chi)(o+) \leq 1-f_1(\chi\_(1-) \leq \mu(\chi) \text{ for } \chi \epsilon X$$

and since $(\mu'_{x_1}) \leq d((\mu'_{x_1}))$ then

$$(\mu'_{x_1})(\chi) \leq 1-f_1(\chi)(o+) \leq f_1(\chi)(1-) \leq \mu(\chi)$$

The same argument can be applied for another fuzzy point, say (μ'_{x_2}), in the fuzzy set μ with $(\mu'_{x_2}) \leq d((\mu'_{x_2})) \leq \mu$, to have an order-preserving fuzzy continuous function $f_2 : X \longrightarrow [0,1]_s$ such that

$$(\mu'_{x_2})(x) \leq 1-f_2(x)(o+) \leq 1-f_2(x)(1-) \leq \mu(x) \text{ for } x \epsilon X$$

We proceed this operation for each fuzzy point in the fuzzy set μ so that, in general case, we have there exist order-preserving fuzzy continuous functions $f_i : X \longrightarrow [o,1]_s$ such that.

$$(\mu'_{x_i})(x) \leq 1-f_i(x)(o+) \leq 1-f_i(x)(1-) \leq \mu(x) \text{ for } x \epsilon X$$

Since μ is an open fuzzy set i.e., μ-sup $\{(\mu'_{x_i})\}$. Then $(X, \leq, \tau)$ is fuzzy completely regular preordered space.

We omit the proof of the following easily established.

Proposition 5.4 A FT_4-preordered space is an $FT_{7/2}$-preordered space.

6- Fuzzy completely normal ordered and FT_5-ordered spaces

Definition 6.1 A fuzzy topological preordered space $(X,\leq,\tau)$ is said to be fuzzy completely normally preordered space iff, for any decreasing (resp. increasing) fuzzy sets μ and,λ in $(X,\leq,\tau)$ with $D(\mu)\leq\lambda$ {resp. $I(\mu)\leq\lambda$} and $\mu\leq 1\text{-}D(1\text{-}\lambda)$ {resp.$\mu\leq 1\text{-}I(1\text{-}\lambda)$}, there exist a decreasing {resp. increasing} τ-open fuzzy set μ_1 and a decreasing {resp. increasing} τ-closed fuzzy set
μ_1 and a decreasing {resp. increasing} τ- closed fuzzy set $\lambda_1$such that

$$\mu\leq\mu_1\leq\lambda_1\leq\lambda$$

If the preordered of the fuzzy completely normally preordered space $(X,\leq,\tau)$ is an order, then $(X,\leq,\tau)$ is called fuzzy completely normally ordered space.

Proposition 6.1 A fuzzy topological preordered space $(X,\leq,\tau)$ is said to be fuzzy completely normally preordered space iff, the following condition holds: Given any decreasing {resp. increasing} fuzzy sets μ, λ with $D(u)\ \leq\lambda$} and $u\ \leq 1\text{-}D(1\text{-}\lambda)$ {resp. $u\leq\ 1\text{-}I(1\text{-}\lambda)$}, there exists a decreasing {resp. increasing} τ-open fuzzy set ν such that $\mu\leq\nu\leq D(\nu)\leq\lambda$ {resp.$\mu\leq\nu\leq I(\nu)\leq\lambda$}.

Proof: Straightfoward.

Proposition 6.2 A fuzzy completely normally preordered space is a fuzzy normally preordered space.

Proof: straightforward.

Remark 6.1 In the case of the discrete order, proposition 6.2 reduces to the result of [6 proposition 24].

Definition 6.2 A fuzzy completely normally preordered space which is also SFT_1 - preordered space is called a FT_5-preordered space.

We omit the proof of the following two easily established.

Proposition 6.3 A FT $_5$-preordered space is a FT_4-preordered space.

Lemma 6.1 A FT_5-preordered space is not productive.

7- Fuzzy perfectly normal ordered and $FT_{11/2}$-ordered spaces

Definition 7.1 A fuzzy topological preordered space $(X,\leq,\tau)$ is said to be fuzzy perfectly normally preordered space if, for every decreasing (resp. increasing) τ-closed fuzzy set μ and a decreasing (resp. increasing) τ-open fuzzy set λ with $\mu\leq\lambda$, there exists on order-preserving fuzzy continuous function $f:X \dashrightarrow [o,1](L)$such that, for each $x\in X$,

$$\mu(x)=1-f(x)(o+)\leq 1-f(x)(1-)=\lambda(x)$$

If the preorder of the fuzzy perfectly normally preordered space $(X,\leq,\tau)$ is an order, then $(X,\leq,\tau)$ is called a fuzzy perfectly normally ordered space.
Proposition 7.1 A fuzzy perfectly normally preordered space is a fuzzy completely normalloy preordered space.

Proof: Let $(X,\leq,\tau)$ be a fuzzy perfectly normally preordered space and let μ,λ be decreasing fuzzy sets with $D(\mu)\leq\lambda$ and $\mu\leq 1-D(1-\lambda)$ Then for a decreasing τ-closed fuzzy set $D(\mu)$ and a decreasing τ-open fuzzy set $(1-I(1-\lambda))$, there exists an order-preserving fuzzy continuous function $f:X \dashrightarrow [o,1]_*$ such that, for each $\chi\in X$,.

$$D(\mu)(X)=1-f(\chi)(o+)\leq 1-f(x)(1-)=1-I(1-\lambda)(\chi).$$

Since f(x) is decreasing, we have.

$$f(x)(1-)\leq f(x)(1+/2)\leq f(x)(1-/2)\leq f(x)(o+)$$

Thus

$$D(\mu)(X)\leq 1-f(X)(1-/2)\leq 1-f(X)(1+/2)\leq 1-I(1-\lambda)(X)$$

Since $1-f(X)(1+/2)=1-\mathbf{R}_{1/2}\ \{f(x)\}$

$$=f^{-1}\{1-\mathbf{R}_{1/2}\}$$

and

$$1-f(X)(1-/2)\ =\mathbf{L}_{1/2}\ f(X)$$

$$=f^{-1}\{\mathbf{L}_{1/2}\}$$

whenever $1-\mathbf{R}_{1/2}$ and $\mathbf{L}_{1/2}$ are, respectively, decreasing closed and open fuzzy sets, and so $f^{-1}(1-\mathbf{R}_{1/2})$ and $f^{-1}(\mathbf{L}_{1/2})$. Thus

$$D(\mu)=f^{-1}(\mathbf{L}_{1/2})\leq f^{-1}(1-\mathbf{R}_{1/2})=[1-I(1-\lambda)]$$

and it follows that.

$$\mu\leq D(\mu)=f^{-1}(\mathbf{L}_{1/2})\ \leq f^{-1}(1-\mathbf{R}_{1/2})=[1-I(1-\lambda)]\leq\lambda$$

Then $(X,\leq,\tau)$ is a fuzzy completely normally preordered space.

Definition 7.2 A fuzzy perfectly normally preordered space which is also SFT_1-preordered space is called a $FT_{11/2}$-preordered space.

Proposition 7.2 A $FT_{11/2}$-preordered space is FT_5-preordered space.

Proof: Straightforward

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الإنتظام والعادية فى الفراغات التوبولوجية الفازية

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الفكرة الرئيسية للبحث هى استحداث بعض مسلمات الانفصال وهى FT_3, $F\ T_{3.5}$, $F\ T_4$, FT_3 ordered spaces وذلك فى الفراغات الفازية المرتبة مع تقديم دراسة لأهم خواص تلك المسلمات .