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COMPACT FUZZY TOPOLOGICAL ORDERED SPACES

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ABSTRACT

The aim of this paper is to define the notion of compactness and local compactness in the area of fuzzy topological and order structures. Also, we study some of their, respective, properties

Keywords: Fuzzy topological ordered spaces, strongly fuzzy Housdorff topological ordered spaces and order-preserving fuzzy continuous functions.

1- INTRODUCTION

In this paper we introduce and study the notions of compactness and local compactness in a fuzzy topological ordered spaces. Section 2 contains some preliminaries. In Section 3 we define the strong closeness of a preorder(resp. order) relation on a fuzzy topological preordered (resp. ordered) spaces. Also, we study the relationship between it and Katsaras's ones[7]. Section 4 is devoted to define and study the notions of compactness in fuzzy topological ordered space. In section 5 we introduce and study the notions of local compactness in fuzzy topological ordered spaces.

2- PRELIMINARIES

The standard basic fuzzy set theoretical and fuzzy topological concepts, used in this paper, can be found in Chang [2], Christoph [3], Guojun [5], Hutton [6], Lowen [8,9,10]. However we recall the following:

DEFINITION 2.1. [4] A fuzzy topological preordered space $(X, \leq, \tau)$ is said to be strongly fuzzy Housdorff topological (SFT_2-) preordered iff, for each pair of fuzzy point p,q with $p \leq q$ (resp. $p \geq q$) in a preordered set $(X, \leq)$, there exist an increasing (resp. decreasing) $\tau-$ open fuzzy set μ containing p and a decreasing (resp. increasing) $\tau-$open fuzzy set λ containing q such that $\mu \wedge \lambda = 0$.

PROPOSITION 2.1 [4] Let f be an order-preserving fuzzy continuous function from a fuzzy topological preordered space $(X, \leq_2, \tau)$ to an SFT_2 −preordered space $(Y, \leq_2, \nu)$. Then $(X, \leq_2, \tau)$ is an SFT_2 −preordered space.

PROPOSITION 2.2 [4] The one-to one open and order-preserving fuzzy continuous function f carries an SFT_2 − preordered space to another SFT_2 − preordered space .

PROPOSITION 2.3 [4] A preordered fuzzy subspace of an SFT_2 −preordered space is again an SFT_2 −preordered space.

PROPOSITION 2.4 [4] Let $\{(X_\alpha, \leq_\alpha, \tau_\alpha) : \alpha \in \mathbf{J}\}$ be an indexed family of SFT_2 −preordered spaces. Then, their product fuzzy topological preordered space $(X, \leq, \tau)$ is an SFT_2 −preordered space.

Throughout this paper we will be concerned with Hutton's compactness [6], namely : a fuzzy topological space is said to be compact if every open cover of a closed fuzzy set has a finite subcover. Hence the constructed compact fuzzy topological preordered (resp. ordered) space reduces to the Hutton's compact fuzzy topological space when its preorder (resp. order) is the discrete order.

3- CLOSED ORDER RELATIONS

In [7] Katsaras showed that the preorder relation in a fuzzy topological preordered space $(X, \leq, \tau)$ is closed iff, the following condition holds : Gived a pair of points $x, y \in X$ with $x \leq y$ (resp. $x \geq y$), there exist neighbourhoods μ and λ of x and y, respectively, such that $i(\mu) \wedge d(\lambda) = 0$ {resp. $d(\mu) \wedge i(\lambda) = 0$}.

Our task in the sequel is to define the closeness of the order relation in a so effective case and we call it a strongly closed order relation.

DEFINITION 3.1. A preorder $\leq$ of a fuzzy topological preordered space $(X,\leq,\tau)$ is called a strongly closed iff, one of the following two equivalent condition holds;

(i) $(X,\leq,\tau)$ is an SFT_2 -preordered space.

(ii) The graph $G(\leq)$ of such preorder is fuzzy closed in $X \times X$.

To justify our usage of the words " strongly closed" we give the following Lemma :

LEMMA 3.1. Strong closeness of a preorder relation, in a fuzzy topological preordered space, implies Katsaras's closeness.

PROOF: Fix a pair x,y of points in a fuzzy topological preordered space $(X,\leq,\tau)$ and consider the fuzzy points μ_x and μ_y with $\mu_x \leq \mu_y$. Strong closeness of the preorder $\leq$ implies that (by Definition 2.1) there is an increasing open τ-fuzzy set μ containing the fuzzy point μ_x and a decreasing τ-open fuzzy set λ containing the fuzzy point μ_y such that $\mu\wedge\lambda=0$. Since $\mu = i(\mu)$ and $\lambda = d(\lambda)$, we have $i(\mu) \wedge d(\lambda) = 0$ for an increasing neighbourhood μ of x and a decreasing neighbiurhood λ of y, then $\leq$ is closed.

But to show that the converse is not true, we give the following Example:

EXAMPLE 3.1. Let $X = \{x,y\}$, $x \leq y$ (resp. $y \leq x$) and consider the fuzzy points $\{\mu_x,\mu_y\}$. Let τ be the fuzzy topology on a preordered set $(X,\leq)$ generated by :

$$\left\{\{\mu_x,\mu_y\}\cup\{\alpha : X \longrightarrow [0,1]\}\right\}.$$

where α is the constant fuzzy set. Then the preorder of such fuzzy topological preordered space $(X,\leq,\tau)$ is closed not strongly closed.

4- COMPACT FUZZY TOPOLOGICAL ORDERED SPACES

A compact fuzzy topological preordered (resp. ordered) space is defined as a compact fuzzy topological space equipped with a strongly closed preorder (resp. order) relation on it.

PROPOSITION 4.1. Let f be a one-to-one open and order-preserving fuzzy continuous function of a compact fuzzy topological preordered space $(X, \leq, \tau)$ onto a fiuzzy topological preordered space $(Y, \leq, \nu)$. Then $(Y, \leq, \nu)$ is compact.

PROOF : By Proposition 2.2, the preorder $\leq$ of $(Y, \leq, \nu)$ is strongly closed. Let μ be a ν −closed fuzzy set with $B \subseteq \tau$ as an open cover of μ i.e., $\mu \leq \sup\{v : v \in B\}$. Then ;

$$\mathbf{f}^{-1}(\mu) \leq \sup\{\mathbf{f}^{-1}(v) : v \in B\}$$

In view of the function $\mathbf{f}$, $\mathbf{f}^{-1}(B)$ is an open cover of τ −closed fuzzy set $\mathbf{f}^{-1}(\mu)$. But $(X, \leq, \tau)$ is compact, then each open cover of $\mathbf{f}^{-1}(\mu)$ has a finite subcover, say,

$$\{\mathbf{f}^{-1}(v_1), \mathbf{f}^{-1}(v_2), \cdots, \mathbf{f}^{-1}(v_n)\}$$

However, since $\mathbf{f}$ is onto, then $\mathbf{f}\{\mathbf{f}^{-1}(B)\}=B$ and

$$\mathbf{f}\{\mathbf{f}^{-1}(v_1), \mathbf{f}^{-1}(v_2), \cdots, \mathbf{f}^{-1}(v_n)\} = v_1, v_2, \cdots, v_n$$

i.e., the open cover B of μ has a finite subcover and, hence $(Y, \leq, \nu)$ is compact.

PROPOSITION 4.2. Any closed preordered fuzzy subspace of a compact fuzzy topological preordered space is compact.

PROOF: Let $(A, \leq_\Lambda, \tau_\Lambda)$ be a closed preordered fuzzy subspace of a compact fuzzy topological preordered space $(X, \leq, \tau)$ and let μ be a τ_Λ − closed fuzzy set with $B \subsetneq \tau$ as an open cover. It follows that μ is $\tau-$closed and there exist a $\tau-$open family B' such that $B = B'/A$. It is clear that B' is an cover of μ in $(X, \leq, \tau)$, such open cover has a finite subcover, say,

$$\{\nu_1, \nu_2, \cdots, \nu_n\}$$

Then it is clear that

$$\{\nu_1 / A, \nu_2 / A, \cdots, \nu_n / A\}$$

is a finite subcover of μ in $(A, \leq_\Lambda, \tau_\Lambda)$. By Proposition 2.3, it follows that the induced preorder $\leq_\Lambda$ is strongly closed and hence $(A, \leq_\Lambda, \tau_\Lambda)$ is compact.

PROPOSITION 4.3. Let $\{(X_\alpha, \leq_\alpha, \tau_\alpha): \alpha = 1, 2, \cdots, n\}$ be a finite family of compact fuzzy topological preordered spaces. Then the product fuzzy topological preordered space $(X, \leq, \tau)$ is, also, compact.

PROOF: Let $\{(X_\alpha, \leq_\alpha, \tau_\alpha): \alpha = 1, 2, \cdots, n\}$ be a finite family of compact fuzzy topological preordered spaces and let $(X, \leq, \tau)$ be their, respective, product fuzzy topological preordered space. In view of Proposition 2., the preorder $\leq$ is strongly closed. Let λ be any $\tau-$closed fuzzy set with B in ;

$$\{P_\alpha^{-1}(\nu_\alpha): \nu_\alpha \in \tau_\alpha, \quad \alpha = 1, 2 \cdots, n\}$$

as an open cover i.e.,

$$\lambda \leq \sup\{\nu: \nu \in B\}$$

Then

$$P_\alpha(\lambda) \leq \sup\{\mu : P_\alpha^{-1}(\mu) \in B\},$$

we choose

$$B_\alpha = \{\mu_\alpha \in \tau_\alpha : P_\alpha^{-1}(\mu_\alpha) \in B\},$$

Then

$$P_\alpha(\lambda) \leq \sup\{\mu_\alpha \in \tau_\alpha : \mu_\alpha \in B_\alpha \text{ and } P_\alpha^{-1}(\mu_\alpha) \in B\},$$

with the projection mappings $P_\alpha : X \longrightarrow X_\alpha$ it follows that the fuzzy set $P_\alpha(\lambda)$ is τ_α – closed. Since $\{(X_\alpha, \leq_\alpha, \tau_\alpha) : \alpha = 1, 2, \cdots, n\}$ are compact fuzzy topological preordered spaces, then each open cover B_α of $P_\alpha(\lambda)$ has a finite subcover, say,

$$\{\mu_1, \mu_2, \ldots, \mu_n\} \subseteq B$$

i.e., the open cover B of λ has a finite subcover. Hence, the product fuzzy topological preordered space $(X, \leq, \tau)$ is compact.

5- LOCALLY COMPACT FUZZY TOPOLOGICAL ORDERED SPACES.

A fuzzy topological space (X, τ) is locally compact [3] if, for each fuzzy point p in X, there exists a compact τ –open fuzzy set μ and a τ –open fuzzy set λ such that $\mu_p < \lambda \leq \mu$.

A locally compact fuzzy topological preordered (resp. ordered) space is defined as a locally compact fuzzy topological space equipped with a strongly closed preorder (resp. order) relation $\leq$ on it.

PROPOSITION 5.1. Let **f** be a one-to-one open and oorder-preserving fuzzy continuous function of a locally compact fuzzy topological preordered space $(X, \leq, \tau)$ onto a fuzzy topological preordered space $(Y, \leq, \nu)$. Then $(Y, \leq, \nu)$ is locally conpact.

PROOF : Since $(X, \leq, \tau)$ has a strongly closed preorder $\leq$, as we have shown in Proposition 3.1 $(Y, \leq, \nu)$ has a strongly closed preorder. Let q be a fuzzy point Y with support x_q and value $0 < t < 1$. Define a fuzzy point p in X by

$$\mu_p(x) = \begin{cases} t & \text{for } \quad x = x_p \\ 0 & \text{otherwise} \end{cases}$$

where $x_p \in \mathbf{f}^{-1}\{x_q\}, x_p$ is the support of the fuzzy point p. Then $\mathbf{f}(p) = q$. By assumption, there is a compact τ –open fuzzy set λ and a τ –open fuzzy set ν

such that $\mu_p < \lambda \leq \nu$. Now the fuzzy point q belongs to $f(\lambda)$ and the fuzzy sets $f(\lambda)$ and $f(\nu)$ are $\nu-$open fuzzy sets with $\mu_q < f(\lambda) \leq f(\nu)$. Let $B \leq \nu$ be an open cover of $f(\lambda)$ i.e.,

$$f(\lambda) \leq \sup\{\sigma : \sigma \in B\},$$

then it is easily shown that

$$\lambda \leq \sup\{f^{-1}(\sigma) : \sigma \in B\},$$

i.e., the family $B_o = \{f^{-1}(\sigma) : \sigma \in B\}$ is an open cover of λ. But λ is compact, then B_o has a finite subcover, say,

$$\{f^{-1}(\sigma_1), f^{-1}(\sigma_2), \ldots, f^{-1}(\sigma_n)\}$$

Since the function **f** is onto, then it is easily to see that ;

$$f\{f^{-1}(\sigma_1), f^{-1}(\sigma_2), \ldots, f^{-1}(\sigma_n)\} = \sigma_1, \sigma_2, \ldots, \sigma_n$$

i.e., the open cover B has $f(\lambda)$ a finite subcover and hence, $(Y, \leq, \nu)$ is locally compact.

THEOREM 5.1. Let $\{(X_\alpha, \leq_\alpha, \tau_\alpha) : \quad \alpha = 1, 2, \cdots, n\}$ be a finite family of locally compact fuzzy topological preordered spaces. Then the product fuzzy topological preordered space $(X, \leq, \tau)$ is, also, locally compact.

PROOF : The preorder of the product fuzzy topological preordered space $(X, \leq, \tau)$ in view of Proposition 3.3, is strongly closed. Let r be a fuzzy point in $X = \prod_{\alpha \in J} X_\alpha$ and let $r_\alpha = \mathbf{P}_\alpha(r)$. By assumption, there is a compact $\tau_\alpha-$open fuzzy set λ_α and a $\tau_\alpha-$open fuzzy set μ_α such that ;

$$\mu_{r_\alpha} < \lambda \leq \mu_\alpha$$

The fuzzy sets $\nu = \inf_{\alpha=1}^{n}\left\{\mathbf{P}_\alpha^{-1}(\mu_\alpha)\right\}$ and $\sigma = \inf_{\alpha=1}^{n}\left\{\mathbf{P}_\alpha^{-1}(\lambda_\alpha)\right\}$ are $\tau-$open and the fuzzy point $r = \inf_{\alpha=1}^{n}\left\{\mathbf{P}_\alpha^{-1}(r_\alpha)\right\}$ satisfies the following

$$\mu_r < \sigma \leq \nu$$

Now, the aim is to show that σ is compact fuzzy set. For this purpose we let

$$B = \{\mathbf{P}_\alpha^{-1}(\nu_\alpha) : \nu_\alpha \in \tau_\alpha , \alpha = 1,2,\ldots,n\}$$

to be an open cover of a τ – open fuzzy set $\sigma = \inf_{\alpha=1}^{n} \left\{\mathbf{P}_\alpha^{-1}(\lambda_\alpha)\right\}$ i.e.,

$$\sigma \leq \sup\{\nu : \nu \in B\}$$

Then

$$\mathbf{P}_\alpha(\sigma) \leq \sup\{\mu_\alpha : \mathbf{P}_\alpha^{-1}(\mu_\alpha) \in B\}$$

we choose the family

$$B_\alpha = \{\mu_\alpha \in \tau_\alpha : \mathbf{P}_\alpha^{-1}(\mu_\alpha) \in B\}$$

Then it follows that

$$P_\alpha(\sigma) \leq \sup\{\mu_\alpha \in \tau_\alpha : \mu_\alpha \in B_\alpha\}$$

Since $\mathbf{P}_\alpha(\sigma) = \inf_{\alpha=1}^{n}\{\lambda_\alpha\}$ is compact, then the open cover has a finite subcover, say,

$$\{\mu_1, \mu_2, \ldots, \mu_n\} \subseteq B_\alpha$$

Then, it follows that

$$\mathbf{P}_\alpha^{-1}\{\mu_1, \mu_2, \ldots, \mu_n\} \subseteq B,$$

and

$$\inf_{\alpha=1}^{n}\{\mathbf{P}_\alpha^{-1}(\mu_\alpha)\} \subseteq B,$$

i.e., the open cover of $\sigma = \inf_{\alpha=1}^{n}\{\mathbf{P}_\alpha^{-1}(\mu_\alpha)\}$ has a finite subcover. Then σ is compact and hence, the product fuzzy topological preordered space $(X, \leq, \tau)$ is locally compact.

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