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ON LOWER SEPARATION AXIOMS

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ABSTRACT

In [3] A. Mashhour et al introduced the concept of fuzzy disjointness. Two fuzzy sets A, B in X are said to be fuzzy disjoint if $A \leq$ co B, where co B is the complement of B. Using this concept, they introduced FT_i separation axioms as a generalization for the basic separation axioms $T_i(i=1,\ldots\ldots,4)$.

In [5] M.K.singal et al used the concepts of regular open fuzzy sets [1] and fuzzy disjointness to introduce some fuzzy almost separation axioms which are stronger than those of Mashhour. Unfortunately, some results in [5] are incorrect.

In this paper, we use the concept of the quasi-coincident relation[4] to restate lower fuzzy separation axioms and give a corrected version for some definitions and results in [5].

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1-Preliminaries

Definition 1.1 [2]. Let X be a non-empty set, $\tau \subset I^X$, the pair (X,τ) is an fts iff it satisfies the following conditions:

(1) $\phi, X \in \tau$

(2) If $u, v \in \tau$, then $u \cap v \in \tau$

(3) If $u_j \in \tau$ for all $j \in J$ then, $\cup\{u_j | j \in J\} \in \tau$

Let A be a fuzzy set in X. The closure and the interior of A(denoted by cl A and int A resp.) are defined by

$cl\ A=\bigcap\{F\in I^X|F\in\tau^c and\ A\leq F\}$ and

$int\ A=\bigcup\{G\in I^X|G\in\tau and\ G\leq A\}$ respectively

Definition 1.2 [1]. Let (X,τ) be an fts. A fuzzy set A in X is said to be regular-open iff A= int cl A. A fuzzy set u in X is said to be regular-closed iff co u is regular-open. Equivalently $u\in I^X$ is regular-closed iff u=cl int u.

Thus every regular-open (regular-closed) fuzzy set is open (closed) but the converse does not hold[1]

Definition 1.3 [1]. A subfamily β of I^X (i.e. $\beta\subset I^X$) is a base for a fuzzy topology on X iff

(1) $X=\bigcup\{B|B\in\beta\}$

(2) For each B_1, B_2 in β and for each fuzzy point $x_\lambda\in B_1\cap B_2$ there exist $B^*\in\beta$ such that $x_\lambda\in B^*\leq B_1\cap B_2$.

Definition 1.4 [1]. Let (X,τ) be an fts. The family $\beta=\{A\in I^X|A$ is regular open $\}$ is a base for a fuzzy topology τ^* on X. The fts (X, τ^*) is called the semi-regularization of (X, τ).The fts(X,τ) is said to be semi-regular space iff $\tau=\tau^*$

Definition 1.5 [4]. Let A,B be two fuzzy sets in X. A is said to be quasi-coincident with B(denoted A q B) iff there exists $x\in X$ such that $A(x)+B(x)>1$. Consequently, a fuzzy point x_λ is quasi-coincident with A iff $\lambda+A(x)>1$.

Definition 1.6 [5]. Let (X,τ) be an fts. A fuzzy set A in X is said to be δ-open iff $A\in\tau^*$ that is $A\in I^X$ is δ-open iff $A=\bigcup_i u_i$ where u_i is a regular open fuzzy set for each i. The fuzzy set A in X is said to be δ-closed iff co A is δ-open. Equivalently, $A\in I^X$ is δ-closed iff $A=\bigcap F_i$ where F_i is a regular-closed fuzzy set for each i.

Proposition 1.1 [1].(1) The closure of an open fuzzy set is regular-closed.

(2) The interior of a closed fuzzy set is regular-open.

Definition 1.7 [1]. Let (X,τ) and (Y,U) be two fts. A mapping f: X---> Y is called a fuzzy almost-continuous mapping iff the inverse image of every regular open fuzzy set in Y is open in X.

Definition 1.8 [5]. Let (X,τ) and (Y,U) be two fts. A mapping f: X---> Y is fuzzy almost open(closed) iff for every regular open (regular-closed) fuzzy set u in X, f(u) is open(closed) in Y.

Lemma 1.1 [5] If f:X---> Y is fuzzy almost continuous and fuzzy almost open, then, the inverse image of every fuzzy regular-open (regular-closed) set is regular-open (regular-closed).

For notions and results used but not defined or shown we refer to [4]

2. Fuzzy quasi-separation Axioms

Using the concept of a quasi-coincident relation, we introduce the fuzzy quasi-separation axioms. These new axioms as well as those in [3] are reduced, in the crisp case to the ordinary basic separation axioms.

Definition 2.1. An fts (X,τ) is said to be:

(i) Fuzzy quasi-T_0 (in short FQT_0) space iff for every pair of fuzzy points x_λ, y_μ in X $(x \neq y)$ there exists $u \in \tau$ such that $x_\lambda\, q\, u \leq co\, y_\mu$ or $y_\mu\, q\, u \leq co\, x_\lambda$

(ii) Fuzzy quasi-T_1 (in short FQT_1) space iff for every pair of fuzzy points x_λ, y_μ in X $(x \neq y)$ there exists $u, v \in \tau$ such that $x_\lambda\, q\, u \leq co\, y_\mu$ and $y_\mu\, q\, v \leq co\, x_\lambda$

(iii) Fuzzy quasi-T_2 (in short FQT_2) space iff for every pair of fuzzy points x_λ, y_μ in X $(x \neq y)$ there exists $u, v \in \tau$ such that $x_\lambda\, q\, u \leq co\, y_\mu$, $y_\mu\, q\, v \leq co\, x_\lambda$ and $u \not{q}\, v$.

(iv) Fuzzy quasi-T_s (in short FQT_s) space iff every fuzzy point in X is closed.

(v) Fuzzy quasi-$T_{2\,1/2}$ (in short FQ$T_{2\,1/2}$) space iff for every pair of fuzzy points x_λ, y_μ in X $(x \neq y)$ there exist $u, v \in \tau$ such that $x_\lambda\, q\, u \leq co\, y_\mu, y_\mu\, q\, v \leq co\, x_\lambda$ and $(cl\ u) \not{q} (cl\ v)$.

(vi) Fuzzy quasi-regular (in short FQR) space iff for every fuzzy point x_λ in X and closed fuzzy set F in X such that $x_\lambda\, q\, (co\, F)$ there exist $u, v \in \tau$ such that $x_\lambda\, q\, u, F\, q\, v$ and $u \cap v = \phi$

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A fuzzy quasi-regular space which is FQT_s is said to be FQT_3-space.

(vii) Fuzzy quasi-normal (in short FQN) space iff for every pair of closed fuzzy sets F_1, F_2 in X such that $F_1 q (coF_2)$ there exist $u, v \in \tau$ such that $F_1 qu, F_2 qv$ and $u \cap v = \phi$

A fuzzy quasi-normal space which is FQT_s is said to be FQT_4-space.

One may notice that, in the crisp case, the above FQT_i axioms are reduced to the ordinary T_i-axiom for every $i \in \{0, 1, 2\,1/2, 3, 4\}$. Also, FQT_s is reduced to the ordinary T_1-axiom.

Theorem 2.1. An fts(X, τ) is FQT_1 iff every crisp point in X is closed.

Proof. Let x_1 be a crisp point in X. Then, for every fuzzy point $p = y_\mu$ in X with $x \neq y$ there exist $u, u_p \in \tau$ such that $x_\lambda qu \leq cop$ and $pqu_p \leq co\, x_1$.

Hence, for every fuzzy point p with p q(co x_1), there exist $u_p \in \tau$ such that p q u_p. Consequently $pq(\cup u_p)$. Thus, p q (co x_1) implies p q$(\cup u_p)$. Since $\cup u_p \leq co\, x_1$, then , for every fuzzy point p with p q $(\cup u_p)$, $pq(co\, x_1)$.
Hence p q ($\cup u_p$) iff p q (co x_1) i.e. $\cup u_p = co\, x_1$.
Therefore, co x_1 is open and x_1 is closed.

Conversely, let x_λ, y_μ be a pair of fuzzy points in X with $x \neq y$. Since x_1 and y_1 are closed; co x_1 and co y_1 are open. Since $x_\lambda \, q \, co\, y_1$; $y_\mu \, q \, co\, x_1$, $co\, x_1 \leq co\, x_\lambda$ and $co\, y_1 \leq co\, y_\mu$; then (X,τ) is FQT_1.

Corollary 2.1. Every FQT_S is FQT_1 but the converse is not true, in general.

Example 2.1. Let $X=\{x_1, x_2\}$ and let $A=\{x_\lambda | x \in X, \lambda \in [1/2,1]\}$ and $B=\{u \in I^X |$ range $u \leq [1/2,1]\}$. Let $\tau = A \cup B \cup \{\phi\}$. Then, τ is a fuzzy topology on X. Since all crisp points are τ-closed; (X,τ) is FQT_1. But the fuzzy point p with support x_1 and value 3/4 is not τ-closed. Consequently, (X,τ) is not FQT_s.

Theorem 2.2. For every $i \in \{0,1,2,2\ 1/2,3,S\}$ the corresponding FQT_i property is hereditary.

Proof. Let us prove, for example, the case i=o. Let (X,τ) be an FQT_0 fts and let (Y,τ_Y) be a subspace of (X,τ). Let x_λ, y_μ be two fuzzy points in Y such that $x \neq y$. Since $Y \leq X$, then, there exist $u \in \tau$ such that $x_\lambda \, q \, u \leq co\, y_\mu$ or $y_\mu \, q \, u \leq co\, x_\lambda$. Let $u^* = Y \cap u \in \tau_Y$, then, $x_\lambda \, q \, u^* \leq co\, y_\mu$ or $y_\mu \, q \, u^* \leq co\, x_\lambda$. Hence, (Y,τ_Y) is FQT_0. The proofs for other cases are similar.

Theorem 2.3. Fuzzy quasi-normality is hereditary with respect to closed subspaces.

Proof. Let (X,τ) be an FQN space and let (Y,τ_Y) be a closed subspace. Let F_1 and F_2 be τ_Y-closed fuzzy sets such that $F_1 \, q \, co_Y F_2$. Since Y is closed, then F_1, F_2 are closed in X and $F_1 \, q \, (co\, F_2)$, then there exist $u, v \in \tau$ such that $F_1 \, q \, u, F_2 \, q \, v$ and $u \cap v = \phi$. Let $u^* = u \cap Y, v^* = v \cap Y$. Hence (Y, τ_Y) is FQN.

Remark 2.1. One may easily prove that for every $i \in \{0,1,2,2\ 1/2,3,4,S\}$ the corresponding FQT_i property is additive (see[3] for the definition of an additive property).

3-Fuzzy almost-quasi-separation axioms

In this section we introduce the concepts of fuzzy almost quasi-separation axioms. Using these new axioms we avoid some deviations in [5].

Definition 3.1. An fts (X, τ) is said to be fuzzy almost-quasi To ($FAQT_0$ for brief) iff every pair of fuzzy points x_λ, y_μ in X with $x \neq y$ there exists a regular open fuzzy set u such that $x_\lambda \, q \, u \leq co \, y_\mu$ or $y_\mu \, q \, u \leq co \, x_\lambda$. Equivalently, there exists a δ-open fuzzy set w such that $x_\lambda \, q \, w \leq co \, y_\mu$ or $y_\mu \, q \, w \leq co \, x_\lambda$.

Remark 3.1. In [5] Singal et al has defined an fts (X, τ) to be FAT_0 if it satisfies one of the following conditions:

(i) There exists a regular-open fuzzy set u in X such that $x_\lambda \in u \leq co \, y_\mu$ or $y_\mu \in u \leq co \, x_\lambda$

(ii) There exists a δ-open fuzzy set W in X such that $x_\lambda \in w \leq co \, y_\mu$ or $y_\mu \in w \leq co \, x_\lambda$.

Indeed, the two conditions are equivalent in the crisp case. But, in general, they are not equivalent. The reason is the known fact that $x_\lambda \in \cup \{u_i \mid i \in J\} \not\Rightarrow$ there exists $i \in J$ such that $x_\lambda \in u_i$, where x_λ is a fuzzy point in X and u_i is a fuzzy set in X for all $i\in J$. Using the concept of a quasi-coincident relation, we avoid this deviation. Now, we give corrected versions for Theorems(4.3)......(4.5) in [5].

Theorem 3.1. An fts(X, τ) is FAQT_o iff for every pair of fuzzy points x_λ, y_μ in X$(x\neq y)$, $x_\lambda \notin \delta\text{-}cl\, y_\mu$ or $y_\mu \notin \delta\text{-}cl\, x_\lambda$.

Proof. Let x_λ, y_μ be a pair of fuzzy points in X with $x\neq y$ then, there exists a regular open fuzzy set u in X such that
$x_\lambda\, q\, u \leq co\, y_\mu$ or $y_\mu\, q\, u \leq co\, x_\lambda$. Let $x_\lambda\, q\, u \leq co\, y_\mu$. Then $x_\lambda \notin co\, u$. Since $y_\mu \leq co\, u$ and co u is δ-closed, then $x_\lambda \notin \delta\text{-}cl\, y_\mu$.

Converseiy, let x_λ, y_μ be a pair of fuzzy points in X with $x\neq y$, $x_\lambda \notin (\delta\text{-}cl\, y_\mu)$ or $y_\mu \notin (\delta\text{-}cl\, x_\lambda)$. Then $x_\lambda\, q\, co(\delta\text{-}cl\, y_\mu)$ or $y_\mu\, q\, co(\delta\text{-}cl\, x_\lambda)$. Since $co(\delta\text{-}cl\, x_\lambda) = \cup_i u_i$, $co(\delta\text{-}cl\, y_\mu) = \cup_i v_i$, where u_i, v_i are regular-open fuzzy sets in X for all i. Then, there exist a regular-open fuzzy set u_i in X such that $x_\lambda\, q\, u_i \leq co\, y_\mu$ or $y_\mu\, q\, u_i \leq co\, x_\lambda$. Thus(X, τ) is FAQ$T_o$.

Corollary 3.1 An fts(X, τ) is FAQT_o iff (X, τ*) is FQT_o.

Corollary 3.2 A fuzzy semi-regular space is FAQT_o iff it is FQT_o.

Definition 3.2 An fts(X, τ) is FAQT_1 iff for every pair of fuzzy points x_λ, y_μ in X with different supports, there exists a pair of regular-open fuzzy sets u,v in X such that $x_\lambda q u \le co\, y_\mu$ and $y_\mu q v \le co\, x_\lambda$.

Theorem 3.2 An fts(X,τ) is FAQT_1 iff every crisp point is δ-closed.

Proof. Let x_1 be a crisp point and let p be a fuzzy point in X such that supp $p \ne x$. Then, there exists regular-open fuzzy set u_p in X such that $p q u_p \le co\, x_1$. Let $A = \cup \{u_p : p q co\, x_1\}$. One may easily verify that $co\, x_1 = A$. Consequently, $co X_1$ is δ-open and x_1 is δ-closed.

Conversely, let x_λ, y_μ be a pair of fuzzy points in X with $x \ne y$, then ,x_1 and y_1 are δ-closed fuzzy sets. Consequently, $co\, x_1$ and $co\, y_1$ are δ-open. Then ,$co\, x_1 = \cup_i u_i$ and $co\, y_1 = \cup_j v_j$ where u_i, v_j are regular-open fuzzy sets in X for all i, j.Then, $y_\mu q (co\, x_1) \le co\, x_\lambda$ and $x_\lambda q (co\, y_1) \le co\, y_\mu$. Hence, there exist two regular-open fuzzy sets u_i, v_j in X such that $y_\mu q u_i \le co\, x_\lambda$ and $x_\lambda q v_j \le co\, y_\mu$. Consequently, (X,τ) is FAQT_1.

Corollary 3.3 An fts(X,τ) is FAQT_1 iff (X,τ^*) is FQT_1

Corollary 3.4 A fuzzy semi-regular space is FAQT_1 iff it is FQT_1

Definition 3.3 An fts (X,τ) is FAQT_s iff every fuzzy point in X is δ-closed.

Thus, every FAQT_s is FAQT_1, but the converse is, in general not true. The fts in example 2.1 is FAQT_1 but it is not FAQT_s.

Definition 3.4. An fts(X,τ) is FAQT_2 space iff for every pair of fuzzy points x_λ, y_μ in X with different supports, there exists a pair of regular-open fuzzy sets u,v in X such that $x_\lambda\, q\, u \leq co\, y_\mu, y_\mu\, q\, v \leq co\, x_\lambda$ and $u \leq co\, v$.

Definition 3.5. An fts(X,τ) is FAQ$T_{2\,1/2}$ space iff for every pair of fuzzy points x_λ, y_μ in X with different supports, there exists a pair of regular-open fuzzy sets u,v in X such that $x_\lambda\, q\, u \leq co\, y_\mu, y_\mu\, q\, v \leq co\, x_\lambda$ and $(\delta\text{-}cl\, u) \leq co\, (\delta\text{-}cl\, v)$. Thus, every FAQ$T_{2\,1/2}$ is FAQT_2.

Theorem 3.3. Let(X,τ) and (Y,U) be fts. let f:X-->Y be an injective fuzzy almost continuous and fuzzy almost open. Then X is FAQT_i if Y is FAQT_i, where i ∈{0,1,2,2 1/2}.

Proof. Let us prove the theorem in case i=0. Let x_λ and y_μ be two fuzzy points in X($x \neq y$). Then f(x_λ) and f(y_μ) are two fuzzy points in Y whose supports are f(x),f(y) respectively, and $f(x) \neq f(y)$. Hence, there exists a regular-open fuzzy set u in Y such that $f(x_\lambda)\, q\, u \leq co\, (f(y_\mu))$ or $f(y_\mu)\, q\, u \leq co\, (f(x_\lambda))$. Therefore, $x_\lambda\, q\, (f^{-1}(u)) \leq co\, (f^{-1}f(y_\mu))$ or

$y_\mu q(f^{-1}(u)) \leq co(f^{-1}f(x_\lambda))$. Since $f^{-1}(u)$ is regular-open, from Lemma 2.1, then $x_\lambda q(f^{-1}(u)) \leq co\, y_\mu$ or $y_\mu q(f^{-1}(u)) \leq co\, x_\lambda$. Thus, (X,τ) is FAQT_o. The proofs of other cases are similar.

Theorem 3.4 Every regular-open subspace of an FAQT_i space is FAQT_i where $i \in \{0,1,2,2\ 1/2\}$.

Proof. We prove the theorem when i=2 . Let (X,τ) be FAQT_2 and let (Y, τ_Y) be a regular-open subspace. Let x_λ, y_μ be two fuzzy points in Y with different supports. Then, there exists two regular-open fuzzy sets u,v in X such that $x_\lambda q u \leq co\, y_\mu$, $y_\mu q v \leq co\, x_\lambda$ and $u \leq co\, v$. Let $u^* = u \cap Y$; $v^* = v \cap Y$. Then u^*, v^* are regular-open fuzzy sets in Y. Thus (Y, τ_Y) is FAQT_2-space. The proofs of other cases are similar.

Corollary 3.4. Every open subspace of an FAQT_i space is FQT_i, where $i \in \{0,1,2,2\ 1/2\}$.

Theorem 3.5. Let (X,τ),(Y,U)be two fts. let f:X--->Ybe an injective and fuzzy almost continuous. Then, (X,τ) is FQT_i if (Y,U) is FAQT_i, where $i \in \{0,1,2,2\ 1/2\}$

Proof. In this case, the inverse image of a regular-open fuzzy set is an open fuzzy set. The proof can be completed using similar arguments as in Theorem 4.3.

Corollary 3.5. An fts(X,τ) is FAQT_i iff (X,τ^*) is FQT_i, where $i \in \{0,1,2,2\ 1/2\}$.

Corollary 3.6. A fuzzy semi-regular space is FAQT_i iff it is FQT_i, where $i \in \{0, 1, 2, 2\,1/2\}$.

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عن مسلمات الانفصال السفلى

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في عام ١٩٨٤ قدم مشهور وآخرون مفهوم التباعد الفازى كالتالى :
يقال أن الفئتين الفازيتين A ,B على الفئة X أنهما متباعدتين فازيا اذا كان co B ⩽ A حيث co B هو مكمله B . وباستخدام هذا المفهوم استطاع مشهور تقديم عدد من مسلمات الانفصال (سميت مسلمات الانفصال FT_i) كتعميم لمسلمات الانفصال الأساسية.

في عام ١٩٩٢ استخدم سنجال وآخرون مفهوم الفئة الفازية المنتظمة (الذى قدمه أزاد فى عام ١٩٨١) ومفهوم التباعد الفازى (الذى قدمه مشهور فى عام ١٩٨٤) واستطاعوا تقديم مسلمات انفصال فازيه أقوى من المسلمات التى قدمها مشهور. ولكنا وجدنا بالدراسة أن بعض النتائج فى بحث سنجال غير صحيحة.
في هذا البحث نستخدم مفهوم علاقة التطابق الظاهرى (التى عرفها منج فى عام ١٩٨٠) لاعادة صياغة مسلمات الانفصال السفلى واعطاء الرؤية الصحيحة لبعض التعريفات والنتائج فى بحث سنجال.