Delta J. Sci. 16 (3) 1992

A THREE POINT BOUNDARY VALUE PROBLEM FOR A THIRD ORDER DIFFERENTIAL EQUATION

BY

*I.A. Gomaa and ** S.Z. Sakr

*Faculty of Engineering, Cairo University.

** Faculty of Engineering of Shoubra,

Zagazig University Benha Branch.

Received: 30-12-1992

ABSTRACT

This paper gives a criterion for the existence and the uniqueness of solutions to three-point boundary value problem associated with a third order differential equation. Also, we have constructed as approximate solution by a quartic spline function.

1. INTRODUCTION

The study of three-point boundary value problems is an interesting area of current research and a great deal of work has been done by many authors in the recent years [3,5,7,8,10].

This paper gives a guarentee for the existence and the uniqueness of solutions of three-point boundary value problems associated with the differential equation.

$$y'''(x) = F(x, y, y', y'')$$

Here the Larry Schauder fixed point theorem [4] is used to prove the existence of a solution, while a separate uniqueness theorem is proved by using the Lipschitz condition.

The solution of boundary value prolems as a rule is not found in closed form, then the methods for their approximate solution assumes a great importance. The present paper proposes and efficient method for finding an approximate solution in the form of a spline function of Fourth degree of a boundary value problem for the differential equations in the form

$$y'''(x) + f(x)\, y''' + g(x)\, y' + r(x)y = p(x)$$

the idea of using spline-functions for an approximate of boundary value problems for differential equations has been applied in a number of researches for instance [1,2,9,11].

2. Existence:

Consider the boundary value problem

$$y'''(x) = F(x, y, y', y'') \qquad (1)$$

subjected to either boundary conditions

$$y(x_1) = y_1 \;,\; y(x_2) = y_2 \;,\; y'(x_2) = m \qquad (2)$$

or

$$y(x_2) = y_2 \;,\; y'(x_2) = m \;,\; y(x_3) = y_3 \qquad (3)$$

The boundary conditions (2) and (3) can be matched [5] to yield a unique solution of (1) satisfying the boundary condition

$$y(x_1) = y_1 \;,\; y(x_2)\; y_2 \;,\; y(x_3) = y_3 \qquad (4)$$

The discussion for the boundary condition (2) is analogous to that of (3), we concentrate on (3). If (1) has a solution satisfying the boundary condition (3), then.

$$y(x) = \int_{x_2}^{x^3} g(x,s)\; F(s, y(s), y'(s), y''(s))\, ds$$

Where g(x,s) is the Green's function

Delta J. Sci. 16 (3) 1992

$$g(x,s) = \begin{cases} -1/2\left(\dfrac{(x-a)(b-s)}{b-a}\right)^2 + 1/2(x-s)^2 & a \le s \le x \le b \\ -1/2\left(\dfrac{(x-a)(b-s)}{b} - a\right)^2 & a \le x \le s \le b \end{cases}$$

In the expression for g(x,s), $a=x_2$, $b=x_3$. Thus we define an operator T on $C^2([x_2, x_3])$ by

$$Ty(x) = \int_{x_2}^{x_3} g(x,s)\, F(s, y(s), y'(s), y''(s))\, ds$$

If $y(x) \in C^2\,[\,x_2, x_3\,]$, the norm is defined by

$$\|y(x)\|_B = \max\left(\sup_{a\le x\le b}|y(x)|, \frac{4}{27}\sup_{a\le x\le b}|y'(x)|, \frac{1}{27}\sup_{a\le x\le b}|y''(x)|\right)$$

where each supremum is taken over the interval $[x_2\ , x_3]$,

Theorem (1): Let $F(x,y,z,\omega)$ be a continuous function in the following domain D:

$D = [a,b] \times Q_1 \times Q_2 \times Q_3$

where

$Q_1 = \{y : |y| \;\; R_1\}$, $Q_2 = \{z : |z| < R_2\}$, $Q_3 = \{\omega : |\omega| \;\; R_3\}$

Furthermore, let $F(x,y,z,\omega)$ be bounded in D, so that there exists a constant M, such that

$F(x, y, z, \omega)| \le M$ (5)

where

$$M < min \{ \frac{81 R_1}{2(b-a)^3}, \frac{6 R_2}{(b-a)^2}, \frac{3 R_3}{2(b-a)} \} \quad (6)$$

Then, there exists a solution y(x) to equation (1) satisfying (2) or (3) and $|y(x)| < R_1$, $|y'(x)| < R_2$, $|y''(x)| < R_3$.

Proof: Let E be the Banach space of continuously differentiable function h(x) ($a \le x \le b$) with the norm.

$$\|h(x)\|_E = max \{ \sup_{a \le x \le b} |h(x)|, \frac{4}{27}(b-a) \sup_{a \le x \le b} |h'(x)|, \frac{1}{27}(b-a)^2 \sup_{a \le x \le b} |h''(x)| \}$$

Consider the sphere

$$S = \{ h(x) : \|h(x)\|_E \le \frac{2M(b-a)^3}{81} \} \text{ of } E$$

Define the operator T on S as follows

$$Th(x) = \int_a^b g(x,s) \; F(s, h(x), h'(s), h''(s)) \; ds$$

put y(x) = Th(x), $a \le x \le b$, then y(x), y'(x), and y"(x) are cotinuous functions of x. From (5) and (6), the function y"(x), is also continuous and its supermum satisfies the condition

$$\sup_{a \le x \le b} |y'''(x)| \le M$$

Let h(x) ∈ S, then we have

$$\max_{a\le x\le b} |y(x)| \le M\int_a^b |g(x,s)|\,ds \le \frac{2M\,(b-a)^3}{81};$$

$$\max|\,y'(x)\,| \le M\int_a^b |g_x(x,s)|\,ds \le \frac{M(b-a)^2}{6};$$

and

$$\max|\,y''(x)\,| \le \int_a^b |\,g_{xx}(x,s)\,ds \le \frac{2M(b-a)}{3}$$

So that T maps S into itself. Let $h_1(x)$, $h_2(x) \in S$, then

$$|y_1(x)-y_2(x)| \le \frac{2\,(b-a)^3}{81}\int_a^b |F(s,h_1,h_1,h_1) - F(s,h_2,h_2,h_2)|\,ds$$

Since F(x,y,z,ω) is continuous, it follows that $|\,y_1(x) - y_2(x)\,| \to 0$ provided that $|\,h_1(x) - h_2(x)\,| \to 0$. Thus T is a continuous operator. For any y(x) in the range of T, y(x) = (Th) (x) for some function h(x), we have $|\,y'''(x)| \le M$, $a\le x\le b$

$$|y(\alpha)-y(\beta)| \le \frac{M(b-a)^2}{6}\,|\alpha-\beta|$$

$$|y'(\alpha)-y'(\beta)| \le \frac{2M(b-a)}{3}\,|\alpha-\beta|$$

Delta J. Sci. 16 (3) 1992

$$|y''(\alpha)-y''(\beta)| \le M|\alpha-\beta|;\ where\ \alpha,\beta \in [a,b]$$

Then the set of functions y(x) in the rage of T are such that y(x), y'(x) and y"(x) are bounded and continuous.

Using Ascoli-Arzela's theorem [4], we conclude that the range of T has a compact closure. Hence Schauder theorem is applicable and T has a fixed point y(x) = Ty(x).

3. **Uniqueness Theorem (2):**

Let the function F(x, y, y',y") be continuous on $[x_1, x_3] \times R^3$, and it also satisfy the Lipschitz condition.

$$|F(x,y_1,z_1,w_1)-F(x,y_2,z_2,w_2)|$$
$$\le(\theta_0|y_1-y_2|+\theta_1|z_1-z_2|+\theta_3|w_1-w_2|)$$

Further, it is assumed that

$$\frac{2\theta_o}{81}h_i^3+\frac{\theta_1 h_i^2}{6}+\frac{2\theta_2 h_i}{3} < 1 \qquad (7)$$

where $h_i = x_{i+1} - x_i$, i=1,2

Proof: We may prove the uniqueness of the solution of equation (1) subjected to (3). Suppose that u(x) and v(x) are two distinct solutions of (1) satisfying (3) such that

$$u(x)=\int_{x_1}^{x_3} g(x,s)\ F(s,u,u',u'')\ ds$$

Delta J. Sci. 16 (3) 1992

$$v(x) = \int_{x_2}^{x_3} F(s, v, v', v'')\ ds$$

Since F(x,y,y',y") satisfies Lipschitz condition, then by using Theorem (1) we obtain

$$\| u(x) - v(x) \| \le \left(\frac{2\theta_o h_2^3}{81} + \frac{\theta_1 h_2^2}{6} + \frac{2\theta_2 h_2}{3}\right) \| u(x) - v(x) \|$$

If

$$\frac{2\theta_o h_2^3}{81} + \frac{\theta_1 h_2^2}{6} + \frac{2\theta 2 h_2}{3} < 1, \textit{then}$$

$$\| u(x) - v(x) \| \rightarrow 0$$

i.e. u(x) = v(x).

Similarly, uniqueness of solution of (1) subjected to (2) can be proved

From Theorem (2) it is seen that if (7) holds, then the hypothesis (i) of Theorems (2.1) and (2.2) in [3] is satisfied. Hence we have

Theorem (3):

Let F(x,y,y',y") be a continuous function satisfying Lipschitz condition. If h_i (i = 1,2) satisfies (7), then the differential equation (1) subjected to (4) has a unique solution.

4. Quartic spline approximation to a three-point boundary value problem for the differential equation

y"'(x) + f(x) y"(x) + g(x)y' + r(x)y = p(x)

The quartic spline $S_j(x)$ interpolating to the function

$y(x)$ at the knots $x_j = x_0 + jh$ $(j = o,1,\ldots, n)$ is given in the interval $x_{j-1} < x < x_j$ by the equation.

$$S_j(x) = M_{j-1}\frac{(x_j-x)^4}{24h} + M_j\frac{(x-x_{j-1})^4}{24h} + (y_{j-1}-\frac{h^3}{24}M_{j-1})(\frac{x_j-x}{h}) + (y_j-\frac{h^3}{24}M_j)(\frac{x-x_j-1}{h}) \quad (8)$$
$$j=1,2,\ldots$$

where $M_j = S'''(x_j)$ and $y_j = y(x_j)$. Hence

$$S_{j+1}(x_j) - \frac{h^2}{8}M_j - \frac{h^2}{24}M_{j+1} + \frac{y_{j+1}-y_j}{h}, \quad (9)$$
$$j=o,1,\ldots,n-1$$

and

$$S_j(x_j) = \frac{h^2}{8}M_j + \frac{h^2}{24}M_{j-1} + \frac{y_j-y_{j-1}}{h}, \quad (10)$$
$$j=1,2,\ldots,n$$

So that the continuity of first derivatives implies

$$\frac{h^2}{24}M_{j-1} + \frac{h^2}{4}M_j + \frac{h^2}{24}M_{j+1} = \frac{y_{j+1}-2y_j+y_{j-1}}{h}, \quad (11)$$
$$j=1,2,\ldots,n-1$$

If we are given the differential equation

$$y''' + f(x)y'' + g(x)y' + r(x)y = p(x) \quad (12)$$

Subjected to the boundary condition

$$y(\alpha) = C_1, \quad y(\beta) = C_2, \quad y(\gamma) = C_3 \quad (13)$$

The approximate solution of (12) in the interval $[\alpha,\beta]$, is analogous the approximate solution in the interval $[\beta,\gamma]$, so we found the approximate solution in $[\alpha,\beta]$.

Let the interval $[\alpha,\beta]$ be divided into n equal subintervals (x_{j-1}, x_j) where $x_j = \alpha + jh$, $j=o,1,..,n$, $x_n = \beta$, and $h= \beta-\alpha/n$, then the requirement that the spline approximation should satisfy the differential equation (12) at the knots x_j ($j= 0,1,..,n$) leads, on using equations (9) and (10) to a set of relationships from which we can eliminate the unknowns $M_0, M_1,..,M_n$. The result, in conjunction with the boundary conditions (13) (in the interval $[\alpha,\beta]$), is a set of tri-diagonal equations for the determination of $y_0, y_1, .., y_n$.

The spline approximation to equation (12), in which case the differential equation gives on using equations (9) and (10).

$$M_j\left(1+\frac{h}{2}f_j-\frac{h^2}{8}g_j\right)-\frac{h^2}{24}g_jM_{j+1}= p_j-r_jy_j-\frac{g_j}{h}(y_{j+1}-y_j), \quad j=0,1,..,n-1 \tag{14}$$

and

$$M_j\left(1+\frac{h}{2}f_j+\frac{h^2}{8}g_j\right)+\frac{h^2}{24}g_jM_{j-1}= p_j-r_jy_j-\frac{g_j}{h}(y_j-y_{j-1}), \quad j=1,2,..,n \tag{15}$$

Equations (14) and (15) constitute 2n equations in the 2n+2 unknowns $M_0, M_1, .., M_n$, and $y_0, y_1, .., y_n$. Elemination of

M_j leads directly to n-1 equations for the unknowns y_0 to y_n which, together with the two boundary conditions, are sufficient for their determination. We note that equations (14) and (15) imply the relations (11).

Addition of equations (14) and (15) gives the relationship

$$\frac{h^2}{24} g_j M_{j-1} + 2\left(1+\frac{h}{2} f_j\right) M_j - \frac{h^2}{24} g_j M_{j+1} = 2(p_j - r_j y_j) - \frac{g_j}{h}(y_{j+1} - y_{j-1}) \quad j=1,2,\ldots,n-1 \tag{16}$$

and elimination of M_j between this equation and equation (11) yields

$$-\frac{h^3}{24}\left(2+hf_j-\frac{h^2}{4} g_j\right) M_{j-1} + \frac{h^3}{24}\left(2+hf_j+\frac{h^2}{4} g_j\right) M_{j+1} + \frac{h^3}{2} p_j = \left(2+hf_j+\frac{h^2}{4} g_j\right) y_{j+1} - 2\left(2+hf_j-\frac{h^3}{4} r_j\right) y_j + \left(2+hf_j-h\frac{2}{4} g_j\right) y_{j-1} \quad j=1,2,\ldots,n-1 \qquad (1$$

But an explicit expression can be obtained for M_{j-1} in terms of y_{j-1} and y_j by eliminating M_j between equation (14) (with j replaced by j-1) and equation (15), namely

Delta J. Sci. 16 (3) 1992

$$A_jM_{j-1} = \left(1+\frac{h}{2}f_j+\frac{h^2}{8}g_j\right)\left(P_{j-1}-y_{j-1}r_{j-1}-\right.$$
$$\left.\frac{g_{j-1}}{h}(y_j-y_{j-1})\right)+ \qquad (18)$$
$$\frac{h^2}{24}g_{j-1}\left(P_j-r_jy_j-\frac{g_j}{h}(y_j-y_{j-1})\right),$$
$$j=1,2,\ldots n$$

where

$$A_j = \left(1+\frac{h}{2}f_{j-1}-\frac{h^2}{8}g_{j-1}\right)\left(1+\frac{h}{2}f_j+\frac{h^2}{8}g_j\right)+ \qquad (19)$$
$$\left(\frac{h^2}{24}\right)^2 g_jg_{j-1}$$

Similarly M_{j+1} can be obtained in terms of y_{j+1} and y_j from equations (14) and equation (15) (with j replaced by j+1)' the resulting expression being.

$$B_jM_{j+1} = \left(1+\frac{h}{2}f_j-\frac{h^2}{8}g_j\right)$$
$$\left(P_{j+1}-r_{j+1}-r_{j+1}y_{j+1}-\frac{g_{j+1}}{h}(y_{j+1}-y_j)\right) \qquad (20)$$
$$-\frac{h^2}{24}g_{j+1}\left(P_j-r_jy_j-\frac{g_j}{h}(y_{j+1}-y_j)\right),$$
$$j=o,1,\ldots,n-1$$

where

$$B_j = A_{j+1} = \left(1+\frac{h}{2}f_j-\frac{h^2}{8}g_j\right)\left(1+\frac{h}{2}f_{j+1}+\frac{h^2}{8}g_{j+1}\right)+$$
$$\left(\frac{h^2}{24}\right)^2 g_jg_{j+1} \qquad (21)$$

Substitution of the expressions for M_{j-1} and M_{j+1} given by equations (18) and (20) into equation (17) leads after straighforward but tedious manipulation to the final three - term recurrence relationship for the spline approximation, namely

$$\begin{aligned}
& y_{j+1}(1+\frac{h}{2}f_j+\frac{h^2}{8}g_j)\ (1+\frac{h}{2}f_j-\frac{h^2}{8}g_j) \\
& \{2+hf_{j+1}+\frac{h^3}{12}r_{j+1}+\frac{h^2}{3}g_{j+1}\}A_j- \\
& y_j\{(2+hf_j-\frac{h3}{4}r_j)\,C_j+\frac{h2}{48}\,(A_jg_{j+1}-B_jg_0j-1) \\
& ((2+hf_j)^2-\frac{h^3}{12}r_j(2+hf_j)-\frac{(h^2g_j)^2}{12})\}+ \\
& y_{j-1}(1+\frac{h}{2}f_j+\frac{h^2}{8}g_j)\ (1+\frac{h}{2}f_j-\frac{h2}{8}g_j) \\
& \{2+hf_{j-1}+\frac{h^3}{12}r_{j-1}-\frac{h^2}{3}g_{j-1}\}= \\
& \frac{h^3}{12}(1+\frac{h}{2}f_j-\frac{h^2}{8}g_j)\ (1+\frac{h}{2}f_j+\frac{h^2}{8}g_j)\ (A_jP_{j+1}+B_jP_{j-1})+ \\
& \frac{h^5}{288}B_jP_jg_{j-1}(1+\frac{h}{2}f_j-\frac{h^2}{8}g_j)- \\
& \frac{h2}{288}A_jP_jg_{j+1}(1+\frac{h}{2}f_j+\frac{h2}{8}g_j)
\end{aligned} \qquad (22)$$

where

$$C_j=2A_jB_j-(\frac{h^2}{24})^2g_j(B_jg_{j-1}+A_jg_{j+1}) \quad (23)$$

Example. Consider the third order differential equation

$$y'''(x)=\frac{22x-10x^3}{(1+x^2)^3}y+\frac{4x^2}{(1+x^2)^2}y'-\frac{x}{(1+X^2)}Y''' \quad (24\text{-a})$$

Subjected to the boundary conditions.

y(0) = 1, y(1) = 0.5 y(2) = 0.2 (24-b)

The function f(x,y,y',y") will not satisfy condtions A [3,5] in the interval (0,2). So, the resuls of Das and Lalli [5] are not applicable to ensure the existence and the uniqueness in this interval. But according to our results, the function

$$F(x,y,y',y'') = \frac{22x-10x^3}{(1+x^2)^3}y + \frac{4x^2}{(1+x^2)^2}y' - \frac{x}{(1+x^2)}y''$$

is continuous and bounded in the domain D, and also satisfies Lipschitz condition with the constants.

$$\theta_o = \max_{0 \le x \le 2} \left| \frac{\partial F}{\partial y} \right| \le 1.5$$

$$\theta_1 = \max_{0 \le x \le 2} \left| \frac{\partial F}{\partial y'} \right| \le 1$$

$$\theta_2 = \max_{0 \le x \le 2} \left| \frac{\partial F}{\partial y''} \right| \le \frac{1}{2}$$

Then, the condition (7) may be reduced to the form

$$\frac{h_i^3}{27} + \frac{h_i^2}{6} + \frac{h_i}{3} < 1 \ , \ i=1,2$$

Thus, the function F(x,y,y',y") satisfies the conditions of the Theorems 1,2 and 3. Hence the differential equation (24) posses a unique solution.

If we divided the interval [0,1] into two equal subintervals, then, from equation (19), (21), and (23).

$A_1 = 1.08$, $B_1 = A_1 = 1.225069444$, $C_1 = 2.646267187$

and from equation (22), we have.

y(0.5) = 0.700720198

Hence from equations (18) and (20), we get

The spline solution to the differential equation (24-a) in the interval [0,1] in given by

$$S(x) = \begin{cases} \frac{2.891756466}{12}x^4 + 2(1/2-x) + 1.371321411x\,, & , x \in (0,1/2) \\ \frac{2.891756466}{12}(1-x)^4 + \frac{0.356537521}{12}(x-1/2)^4 + 1.371321411(1-x) + 0.996286067(x-1/2)\,, & , x \in [1/2,1] \end{cases}$$

Similarly if we divide the interval [1,2] into two equal subintervals the spline solution is given by

$$S(x) = \begin{cases} \frac{0.388981119}{12}(3/2-x)^4 - \frac{0.22665869}{12}(x-1)^4 + 0.995948113(3/2-x) + 0.699612548(x-1)\,, & x \in [1,3/2] \\ -\frac{0.22665869}{12}(2-x)^4 - \frac{0.2308815}{12}(x-3/2)^4 + 0.699612548(2-x) + 0.402405015(x-3/2)\,, & x \in [3/2,2] \end{cases}$$

Delta J. Sci. 16 (3) 1992

REFERENCES

1- Ahlberg, Nilson, and Walsh. "The theory of splines and their applications" Academic press, New York (1967)

2- Albasiny and Hoskins "Cubic spline solutions to two-point boundary value problems", Computer Journal Vol. (12) (1968) pp.151-153.

3- Barr and Sherman,. Existence and uniqueness of solutions of three-point boundary value problem." Journal of differential equation vol. (13), (1973) pp. 197-212.

4- Courant and Hilbert, " Methods of mathematical physies " New York, vol. 1 (1961) page 59, Vol. 2 (1962) page 357.

5- Das and Lalli, "Boundary value problems for y"'(x) = F(x,y,y")" Journal of mathematical analysis and application vol. 81, (1981), pp. 300-307.

6- Hartman " Ordinary differential equations" . John Wiley & Sons (1964).

7- Jackson and Schrader, "Existence and uniqueness of solutions of boundary value problems for third order differential equations" Journal of differential equations, vol. 9 (1971) (46-54).

8- Jockson "Existence and uniqueness of solutions of boundary value problems for third order differential equations" Journal of differential equations Vol. 13 (1973) (432-437).

9- James L. Blue, "Spline function methods for nonlinear boundary value problems",, Comm. ACM. Vol. 12 No 6 (1969) pp. 327-329.

10- Marty and prasad " Three-point boundary value problem Existence and uniquueness" Yokohama mathematical Journal Vol. 29 (1981) (101-105).

11- Nikolova and Bainov "Application of spline-functions for the construction of an approximate solution of boundary value problems for a class of functional differential equations" Yokohama mathematical Journal Vol. 29(1981) (107-122).

Delta J. Sci. 16 (3) 1992

المسأله الحديه ذات النقاط الثلاث لمعادلة تفاضليه من الرتبه الثالثه

* د. إبراهيم أحمد جمعه   * * صبحى زكريا صقر
* أستاذ م الرياضيات - كلية الهندسة - جامعة القاهرة
* * قسم الرياضيات - كلية الهندسة بشبرا - جامعة الزقازيق - فرع بنها

هذا البحث يتعرض لدراسة وجود ووحدانية الحل لمسألة حديه ذات نقاط ثلاث لمعادله تفاضليه عاديه من الرتبه الثالثه بإستخدام نظرية ليرى شودر وشرط ليبشتز ،

تستخدم نظرية ليرى شودر فى إثبات وجود الحل بينما يستخدم شرط ليبشتز فى إثبات وحدانيه الحل ،

كذلك يتعرض البحث لتكوين حل تقريبى للمعادلة الخطيه ثلاثية النقط من الرتبه الثالثة بواسطة دوال الأسبلين ذات الدرجة الرابعة ،