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ON THE INFINITY NORM OF THE INVERSE OF DIAGONALLY DOMINANT MATRICES

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ABSTRACT

In this paper we give proofs concerning estimates of the infinity norm of certain type of matrices which are used throughout the numerical applications to many problems. Also bounds of the elements of the L and U of the LU decomposition of such matrices are given. In addition, we show how these results are useful in the numerical applications.

1- INTRODUCTION

Diagonally dominant matrices appear frequently in many applications of the numerical techniques to ordinary and partial differential equations. In fact resulting systems of equations involve always such matrices, and it is necessary to have a measure to the size of these matrices in order to estimate the error bounds of approximating the derivatives in these differential equations. The maximum (infinity) norm is

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the most commonly used measure in such cases. The maximum (infinity) norm is the most commonly used measure in such cases. The value of this norm is to be evaluated according to the problem, but in most physical and engineering applications the resulting matrices use to be bounded and in many cases tridiagonal, such as in [1], [2], [4] and many others. Our aim here is to find an easy way to estimate such norm by showing that the infinity norm of these matrices is almost equal to the L_2- Norm (which is easy to be evaluated) whenever the dimension of the matrix becomes large. This is the case when the stepsize of the numerical scheme is chosen small enough in order to get certain accuracy. In our work we prove this relation over the range of a certain type of matrices. The relation helps in estimating error bound of the interpolating polynomials and their derivatives over the knot points (see [2], [4]). Here we concentrate on tridiagonal diagonally dominant martices which is a common feature in numerical applications.

2. The infinity norm of the inverse of a tridiagonal diagonally dominant matrix.

We shall consider the following two cases:

Case 1: A Symmetric matrix A with constant elements, so first perform the *LU* decomposition for A, then give some bounds of the elements of the matrices *L* and *U* which in turn in getting an estimate for $\|A^{-1}\|$. In this case, without loss of generality, we can consider A with the elements on the two subdiagonals to be unity, if this is not the case a common factor can be taken of each row of the matrix. Thus let

$$A=\begin{pmatrix} k & 1 & 0 & & & \\ 1 & k & 1 & & & \\ 0 & 1 & k & 1 & & \\ & & & 1 & k & 1 \\ & & & & 1 & k \end{pmatrix} \qquad (1)$$

and A = *LU*
where

$$L=\begin{pmatrix} 1 & & & \\ l_2 & & & \\ & l_3 & & \\ & & l_n & 1 \end{pmatrix}, U=\begin{pmatrix} u_1 & 1 & & \\ & u_2 & 1 & \\ & & & 1 \\ & & & u_n \end{pmatrix}$$

Comparing the elements of *LU* and A we get the following recurrence relations for u_n and l_n.

$$u_1=k \quad , \quad u_n=k-\frac{1}{u_{n-1}} \quad and$$

$$l_2=\frac{1}{k} \quad , l_n=\frac{1}{k-l_{n-1}} \quad , n=3(1)N$$

consider the first of these relations

$$u_n=k-\frac{1}{u_{n-1}} \quad ,u_1=k \text{ and } n=2(1)N \tag{2}$$

this is a fixed point iteration for the function

$$f(u) = u^2 - ku + 1 = 0 \tag{3}$$

with the iteration function

$$g(u)=k-\frac{1}{u}, (u\neq 0)$$

the condition of convergence is satisfied in some interval around the root as

$|g'(u_i)| < 1$ whenever $|k| > 2$

which is true since A is diagonally dominant. So the iteration formula (2) converges to a root of $f(u)$. Next it is useful to determine the range of u_n around this root in which u_n converges, this is given by the following lemma.

Lemma 1: The iteration values u_n , $n = 1(1)N$ given by relation (2) for equation (3) satisfy

$$\frac{k}{2}+\frac{1}{2}\sqrt{k^2-4}<u_n\leq k, k>2 \tag{4}$$

$$k \le u_n < \frac{k}{2} + \frac{1}{2}\sqrt{k^2-4}, k < -2 \qquad (5)$$

proof:

Let k > 2 relation (4) can be proved by induction

$$u_1 = k > \frac{k}{2} + \frac{1}{2}\sqrt{k^2-4}$$

Suppose

$$u_{N-1} > \frac{k}{2} + \frac{1}{2}\sqrt{k^2-4}$$

$$u_N = k - \frac{1}{u_{N-1}} > \frac{k}{2} + \frac{1}{2}\sqrt{k^2-4}$$

hence the left hand side of (4) is true also it can be easily shown that

$$u_n - u_{n-1} < 0$$

i.e. u_n is monotonic decreasing, i.e. bounded above by k. Thus inequality (4) is proved. Similarly inequality (5) can be proved to be true, hence the proof of the lemma.

Lemma 2: The sequence l_n, $n = 2(1)N$ satisfying the recurrence relation

$$l_n = \frac{1}{k - l_{n-1}}$$

satisfies the inequalities

$$\frac{1}{k} \le l_n < \frac{2}{k+\sqrt{k^2-4}}, k>2 \qquad (6)$$

$$\frac{2}{k+\sqrt{k^2-4}} < l_n \le \frac{1}{k}, k<-2 \qquad (7)$$

Proof:

Since $l_n \, u_{n-1} = 1$

$$i.e \; l_n = \frac{1}{u_{n-1}}$$

the lemma is true from Lemma 1.

Now if we consider the system of equations

$$A\,X = Z \text{ with } \|Z\|_\infty = 1$$

$$i.e \; LY = Z \text{ and } Y = Ux \dashrightarrow Y_1 = Z_1$$

$$Y_n = Z_n - l_n \, Y_{n-1}, \quad n = 2(1)N$$

from which we get

$$|Y_n| \le 1 + |l_n||y_{n-1}| \qquad (8)$$

Lemma 3: $|Y_n|$ as given by inequality (8) satisfies

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$$|Y_n| < (k-2 + \sqrt{k^2-4})/2k-4, \quad k>2,$$

$$|Y_n| < (k + 2 + \sqrt{k^2-4})/2k+4, \quad k<-2$$

Proof

$$|y_n| \le 1 + |l_n||y_{n-1}|$$

$$\le 1 + |l_n|(1+|l_{n-1}||y_{n-2}|)$$

but from lemma 1

$$|l_n| < \frac{2}{k+\sqrt{k^2-4}} = \frac{2}{P}, \; P = k+\sqrt{k^2-4};$$

therefore

$$|y_n| < 1+\frac{2}{p}(1+\frac{2}{p}+\ldots)$$

$$< \frac{1}{1-2/p} = \frac{p}{p-2}$$

$$|y_n| < (k-2+\sqrt{k^2-4})/(2k-4)$$

Similarly the second case when k < -2 can be proved.

Lemma 4: The N components of the vector $\underline{x}$ satisfying

$$U\underline{x} = \underline{y}$$

satisfy the inequality

$$|x_n| < \frac{1}{|k|-2},\ |k|>2,\ n=1(1)N$$

Proof:

Let k > 2, and by applying the back substitution technique to the system of linear equations , we get

$$x_N = y_N/u_N$$

and $x_n = (y_n - x_{n+1})/u_n$, $n = N-1, N-2, \ldots, 1$

$$|x_N| \le \frac{\max_n |y_n|}{\min_n |u_n|}$$

$$< \frac{k-2+\sqrt{k^2-4}}{2k-4} \cdot \frac{1}{1/2\,(k+\sqrt{k^2-4})} \quad \textit{from Lemmas } 1,2$$

$$< \frac{1}{k-2}$$

So it is true for n = N.

For n = $N-1$, $N-2$, ..., 1 we have

$$|x_n| \le \frac{|y_n| + |x_{n+1}|}{\min_n |u_n|}$$

Suppose the inequality holds for n + 1 , thus

$$|x_{n+1}| < \frac{1}{k-2}$$

$$|x_n| \le \frac{|y_n| + (1/k-2)}{1/2\,(k+\sqrt{k^2-4})}$$

Using Lemma 3, we get

$$|x_n| < \frac{1}{k-2}$$

This means that the case is true for *n*, i.e. the inequality is true for all values of *n*. Similarly it can be proved for $k < -2$, hence the proof of the Lemma.

3. The infinity norm of the inverse of a matrix A

Let

$A\,\underline{x} = \underline{z}$ i.e. $\underline{x}\ A^{-1}\ \underline{z}$

where A is diagonally dominant symmetric matrix

i.e.

$$A = [a_{ij}] = \begin{cases} k & i=j \\ 1 & |i-j|=1 \\ 0 & otherwise \end{cases}$$

also let $A = LU$

Thus

$$\|A^{-1}\|_\infty = \max_{\|z\|_\infty = 1} \|A^{-1}z\|_\infty = \max_{\|z\|_\infty = 1} \|x\|_\infty < \frac{1}{|k|-2}$$

from Lemma 2

but it is known that

$$\|A^{-1}\|_\infty \geq \|A^{-1}\|_2$$

Since A is symmetric

$$\|A^{-1}\|_2 = \frac{1}{\min\ e.v.\ (A)}$$

The eigenvalues of A as given by Smith [5] are:

$$\lambda_s = |k| + 2\cos\frac{s\pi}{N+1},\quad s = 1(1)N$$

So

$$\min_s \lambda_s = |k| + 2\cos\frac{N\pi}{N+1}$$

therefore

$$\frac{1}{|k|+2\cos(N\pi/N+1)} \leq \|A^{-1}\|_\infty \leq \frac{1}{|k|-2} \text{ -----}(*)$$

Thus when N becomes sufficiently large

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$$\|A^{-1}\|_\infty = \|A^{-1}\|_2$$

Hence the following theorem has already been proved.

Theorem: Let A be a tridiagonal symmetric diagonally dominant matrix of order $N \times N$, then as N becomes large, the infinity and L_2 -norms for the matrix A^{-1} tend to the same value i.e.

$$\|A^{-1}\|_\infty \sim \|A^{-1}\|_2$$

for N sufficiently large.

By expanding the cosine in (*), it can be shown that

$$\|A^{-1}\|_\infty \sim \|A^{-1}\|_2 + 0(1/N^2)$$

Case II. A is nonsymmetric diagonally dominant with variable coefficients

Let

$$A = \begin{pmatrix} b_1 & c_1 & & & \\ d_2 & b_2 & c_2 & & \\ & & & & \\ & & & & c_{N-1} \\ & & & d_N & b_N \end{pmatrix}$$

where

b_i, c_i, d_i are functions of x and $|b_i| - \{ |d_i| + |c_i| \} > 0$

If A $\underline{x}$ = $\underline{z}$

$$\|A\|_\infty = \sup_{\|x\|_\infty = 1} \|Ax\|_\infty$$

$$\|Ax\|_\infty \geq |\sum_k a_{jk}\, x_k| j = 1,2,\ldots,N$$

$$= |d_j x_{j-1} + b_j x_j + c_j x_{j+1}|$$

$$\geq |b_j x_j| - (|d_j x_{j-1}| + |c_j x_{j+1}|)$$

Let $$|x_j| = \max_j |x_j|,$$

then we have established the following inequality

$$\|A\, x\|_\infty \geq E_j |X_i|, \quad E_j = |b_i| - (|d_j| + |c_j|)$$

$$x = A^{-1}\, z$$

$$\|A^{-1}\|_\infty = \sup_{\|z\| \neq 0} \frac{\|A^{-1} z\|_\infty}{\|z\|_\infty} = \sup_{\|x\| \neq 0} \frac{\|x\|_\infty}{\|A\, x\|_\infty} \leq \frac{1}{\min_j E_j}$$

$$\leq \frac{1}{\min_j E_j}$$

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$$i.e.\ \|A^{-1}\|_\infty \le \frac{1}{\min_j(|b_j| - (|d_j| + |c_j|))}$$

this last inequality gives bound on $\|A^{-1}\|$

In the previous special case when

$b_j = k$, $c_j = d_j = 1$

$$\|A^{-1}\|_\infty \le \frac{1}{|k|-2}$$

which is the same result obtained in Case I.

Also even in the worst case for the norm bound when

$\underline{x} = (-1,1,-1, \ldots)^T$

$$\| A \underline{x} \|_* = k - 2$$

and so

$$\| A^{-1} \|_* = 1/k - 2$$

4- Application to two point boundary value problems

In this section we are not going in details but jus adopt the work shown in [1] , [4].

Consider the two point boundary value problem

$$p(x)y'' + q(x)y' + r(x)y = f(x,y(x)), \quad x \in [a,b]$$

$$y(a) = \alpha$$

$$y(b) = \beta$$

on applying finite difference scheme or collocation scheme b considering the solution as

$$y(x) = \Sigma_i c_i B_i(x)$$

$B_i(x)$ are spline functions, c_i are constants leading to th system of equations (collocating as the grid points) $A\,\underline{c} = \underline{f}\,(\underline{c})$ to be solved and in which $\| A^{-1} \|_{\infty}$ is needed to get an error bound for the solution.

In case of cubic B-splines defined as :

$$B_i(x) = \frac{1}{h^3} \begin{cases} (x-x_{i-2})^3 & , x_{i-2} \le x \le x_{i-1} \\ h^3+3h^2(x_{i-1}-x)+3h(x-x_{i-1})^2-3(x-x_{i-1})^3 & , x_{i-1} \le x \le x_i \\ h^3+3h^2(x_{i+1}-x)+3h(x_{i+1}-x)^2-3(x_{i+1}-x)^3 & , x_i \le x \le x_{i+1} \\ (x_{i+2}-x)^3 & , x_{i+1} \le x \le x_{i+2} \\ 0 & , \text{Other wise} \end{cases}$$

For which the matrix A becomes : ($\alpha = o$, $\beta = 0$)

$$A=\begin{pmatrix} b_0 & c_0 & 0 & 0 & 0 \\ d_1 & b_1 & c_1 & 0 & 0 \\ 0 & d_2 & b_2 & c_2 & 0 \\ 0 & 0 & d_{n-1} & b_{n-1} & c_{n-1} \\ 0 & 0 & 0 & d_n & b_n \end{pmatrix}$$

$$\text{where } b_0 = \frac{-36p(x_0)}{h^2}+\frac{12q(x_0)}{h},\quad c_0 = \frac{6q(x_0)}{h}$$

$$d_i=\frac{6p(x_i)}{h^2} - \frac{3q(x_i)}{h} + r(x_i),\; b_i=\frac{-12p(x_i)}{h^2} + 4r(x_i),$$

$$c_i=\frac{6p\ (x_i)}{h^2} + \frac{3q\ (x_i)}{h} + r(x_i)$$

$i = 1,2, \text{--}, n - 1$ and

$$d_n = \frac{-6q\ (x_n)}{h}\ ,c_n = \frac{-36p\ (x_n)}{h^2} - \frac{12q(x_n)}{h}$$

with h sufficiently small, the matrix A is diagonally dominant, provided that $p(x_i)\ r(x_i) < 0$, $i = 0(1)n$. In this case the maximum norm of the inverse matrix A^{-1} as used in [1], [4] satisfies the inequality

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$$\|A^{-1}\|_\infty \le \frac{1}{6 \min_i |r(x_i)|}$$

which agrees with present work, also this result was used in other work on partial differential equation.

DISCUSSION AND CONCLUSION

The infinity norm is the widely used for error bounds in numerical analysis, we have shown here how to get an estimate of this norm for the inverse of some matrices, making use of the well known form of the L_1 - norm of these matrices. Morever it should also be noted in this work, through the lemmas shown, that some useful properties of the matrices L and U (A = LU) are obtained, in addition to the bounds shown on their elements which will help in the error analysis of different numerical applications.

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دراسة للمقياس اللانهائى لمعكوسات المصفوفات القطرية الحاكمه

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فى هذا البحث تم إجراء تقدير المقياس اللانهائى لنوع معين من المصفوفات التى تستخدم من خلال تطبيقات التحليل العددى للعديد من المسائل الرياضيه • أمكن كذلك حساب حدود الخطأ لناتج تحليل مثل هذا النوع من المصفوفات • إضافه إلى ذلك فقد بينا فوائد النتائج التى توصلنا إليها فى التطبيقات العدديه •