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D-EQUIVALENT TOPOLOGIES

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ABSTRACT

In this paper we define two topologies τ and $\mathcal{U}$ on a set X to be D-equivalent iff the class of nowhere dense subsets of X with respect to τ is precisely the class of nowhere dense with respect to $\mathcal{U}$. Equivalence in the sense of Levine [7] and D-equivalence are concided for topologies with the same classes of semi-open sets. Some characterizations and properties of D-equivalent topologies are obtained. We investigate the D-equivalence of a topology τ and some of its contractions (expansions). Also D-equivalent topologies which have the same α-sets and the same regular open sets are investigates.

1- INTRODUCTION

Throughout the present paper τ and $\mathcal{U}$ are two topologies on a set X on which no separation axioms are assumed unless explicitly stated. cl_τ (resp. int_τ) and cl_u (resp. int_u) denote closure (resp. interior) operators with respect to τ and $\mathcal{U}$. In [13], [10], [9], α-open, preopen and semiopen sets were introduced respectively. $\alpha O(X,\tau)$ (resp. $PO(X,\tau)$, $SO(X,\tau)$) is the coresponding classes of these types of sets. The complements of the last sets are α-closed, preclosed (denoted by $PC(X,\tau)$) and semiclosed, respectively. The class of α-open forms a

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topology denoted by τ^{α} [13]. The class of all regular open sets (denoted by $RO(X,\tau)$) is a base for a topology τ_s called the semiregularization of τ [15]. If τ' is a topology on X, and $RO(X,\tau) = RO(X,\tau')$, then τ' is ro-equivalent to τ [5]. A topology τ on X is a D-topology [8] if every non-empty open set is dense in X. A space X is called α-compact [11] if every α-open cover of X has a finite subcover. X is semi-T_2' [1] if for each $x,y \in X$, $x \neq y$, there exist U and V $\in SO(X,\tau)$ such that $x \in U$, $y \in V$, and $cl(U) \cap cl(V) = \phi$. A subset S of X is called an RS-compact relative to X [14] if for every cover $\{V_i : i \in I\}$ of S by regular closed sets of X, there exists a finite subset I_o of I such that $S \subset \bigcup \{int(V_i) : i \in I_o\}$. If $R'(\tau) = \{U \in \tau : X - U$ is RS-compact relative to $\tau\}$, then $R'(\tau)$ is a base for a topology $R(\tau)$ on X, called the CO-RS-compact topology on X [2]. (X,τ) is resolvable [6], if there is a subset D of X such that D and X-D are both dense in X. A space X is irresolvable if it is not resolvable. If $\mathcal{F}$ is a filter on X, then the topology $\tau(\mathcal{F}) = \{U \cap F : U \in \tau, F \in \mathcal{F}\}$ is called a filter extension of τ [3]. In [7] the concept of equivalence in the sense of Levine have been introduced as follows "Let τ and $\mathcal{U}$ be two topologies on a set X, we say that τ and $\mathcal{U}$ are equivalent iff (X,τ) and $(X,\mathcal{U})$ have identical dense sets." One easily can deduce that the complement of a nowhere dense subset (denoted by nwd) is dense, and the converse is not true, in general as shown by the following example.

Example 1.1. Let $X = \{a,b,c\}$, and $\tau = \{\phi, X, \{a\}, \{b,c\}\}$, then we notice that $\{a,b\}$ and $\{a,c\}$ are dense sets but their complements are not nowhere dense sets.

Lemma 1.1. (a) If A is dense and open subset of a space X, then (X-A) is nowhere dense

(b) If $A \subset B \subset X$, and A is dense, then B is dense.

(c) If $A \subset B \subset X$, and B is nowhere dense, then A is also nowhere dense.

Proof. Obvious.

2. D-EQUIVALENT TOPOLOGIES

Definition 2.1 Let τ and $\mathcal{U}$ be two topologies on a set X, we say that τ and $\mathcal{U}$ are D-equivalent iff (X, τ) and $(X, \mathcal{U})$ have identical nowhere dense sets.

Remark 2.1: The class of equivalent topologies in the sense of Levine are proper subclass of the class of D-equivalent topologies. The following example ensures that they are not identical.

Example 2.1. Let $X = \{a,b,c,d\}$ and $\tau = \{X, \phi, \{a\}, \{b\}, \{a,b\}, \{b,c,d\}\}$, and $\mathcal{U} = \{X, \phi, \{a,b\}, \{a,b,d\}, \{a,b,c\}\}$. We notice that τ and $\mathcal{U}$ are D-equivalent but not equivalent in the sense of Levine.

In the following we give characterizations and properties of D-equivalent topologies.

Theorem. 2.1. If τ and $\mathcal{U}$ are two topologies on X. Then the following are equivalent:

i) τ and $\mathcal{U}$ are D-equivalent topologies.

ii) For each $A \in \tau \cap \mathcal{U}$, $int_u cl_\tau A \subset cl_u A$, and $int_\tau cl_u A \subset cl_\tau A$.

iii) For each $A \in \tau \cap \mathcal{U}$, A is dense in τ iff A is dense in $\mathcal{U}$.

Proof. (i)$\Rightarrow$(ii) We assume that $int_\tau\ cl_u A \not\subset cl_\tau A$, then $\Phi \neq int_\tau\ cl_u A \cap (X - cl_\tau A) = int_\tau\ (cl_u A \cap (X-A)) \subset int_\tau cl_\tau (cl_u A \cap (X-A))$. Hence $(cl_u A \cap (X-A))$ is not nowhere dense in τ, and $\Phi \neq int_u cl_u (cl_u A \cap (X-A)) \subset int_u (cl_u A \cap cl_u (X-A)) = int_u cl_u A \cap int_u cl_u (X-A) \subset cl_u A \cap (X - cl_u A) = \Phi$. This is a contradiction.

(ii)$\Rightarrow$(iii) Let $A \in \tau \cap \mathcal{U}$, A is dense in τ, then $int_u cl_\tau A = X \subset cl_u A$. Hence A is dense in $\mathcal{U}$.

(iii)$\Rightarrow$(i) Let A be a nowhere dense in τ, then $(X - cl_\tau A)$ is dense and open in τ. Hence $(X - cl_\tau A)$ is dense and open in $\mathcal{U}$. Thus $cl_\tau A$ is nowhere dense in $\mathcal{U}$, and A is nowhere dense in $\mathcal{U}$.

Corollary 2.2. If τ and $\mathcal{U}$ are D-equivalent then, $int_u cl_\tau A \subset cl_u A$ (resp. $int_\tau\ cl_u A \subset cl_\tau A$), for each $A \in \tau$ (resp. $A \in \mathcal{U}$)

COROLLARY 2.3. τ and $\mathcal{U}$ are D-equivalent topologies on X iff for each $A \subset X$, $int_\tau\ cl_\tau\ A \neq \Phi$ iff $int_u cl_u A \neq \Phi$.

THEOREM 2.4. If τ and $\mathcal{U}$ are D-equivalent topologies on X, then each proper τ-preclosed (resp. $\mathcal{U}$-preclosed) subset contains a proper $\mathcal{U}$-open (resp. τ-open) subset.

Proof. Let $\Phi \neq U \in PC(X,\tau)$, then $(X-U)$ is preopen in (X,τ), and hence is not nowhere dense in (X,τ). Thus it is not nowhere dense in $(X,\mathcal{U})$, and $\Phi \neq int_u cl_u (X-U) \subset cl_u (X-U) = X - int_u U$. Hence $int_u U \neq \Phi$, and $\Phi \neq int_u U \subset U$.

Lemma 2.5. Topologies which have the same α-open sets are D-equivalent.

Proof. By Proposition (5) in [13], the proof is obvious.

Lemma 2.6 If X is a space with two topologies τ, τ' such that $\tau \subset \tau' \subset \tau^{\alpha}$. Then τ, τ' and τ^{α} are D-equivalent.

Proof. By Proposition (10) in [13] and Lemma (2.5), the proof is obvious.

Lemma 2.7 If (X,τ) is a space, and $(X,R(\tau))$ is a semi-T'_2. such that $int_{\tau}\, G = int_{R(\tau)} G$ for each $G \in \tau^{\alpha}$. Then $R(\tau)$ and τ are D-equivalent.

Proof. By Lemma (3.4) in [2], and Lemma (2.5), the proof is obvious

Theorem 2.8 . If $\tau \subset \tau'$ such that τ, τ' are D-equivalent, and τ'^{α} is α-compact space. Then

(a) $\tau^{\alpha} \subset \tau'^{\alpha}$ (b) τ, τ'^{α}, and τ^{α} are α-compact.

Proof. (a) Let $A \in \tau^{\alpha}$, then $A = U - N$ [13] where $U \in \tau$, and N is nowhere dense in τ . But $\tau \subset \tau'$, and τ , τ' are D-equivalent, then $U \in \tau'$, and N is nowhere dense in τ'. Thus $A \in \tau'^{\alpha}$.

(b) By (a), the proof is obvious.

Theorem 2.9. A topology τ on X is a D-topology iff the complement of any open set is nowhere dense in X.

Proof. Let $\Phi \neq A \in \tau$, then $int_{\tau}\ cl_{\tau}\ (X-A) = \Phi$, and $cl_{\tau}\ A = X$. Thus (X, τ) is a D topology. Conversely we assume that (X, τ) is a D-topology, and $\Phi \neq A \in \tau$, then $cl_{\tau}\ A = X$. Hence $int_{\tau}\ cl_{\tau}\ (X-A) = \Phi$.

Corollary 2.10 A topology τ on X is a D-topology iff every non empty closed set is nowhere dense in X.

3. SUBSPACES AND SOME OTHER PROPERTIES

Lemma 3.1. If A is a subspace of (X, τ), and S is nowhere dense in A, then S is nowhere dense in X.

Proof. Let S be not nowhere dense in τ, then $int_{\tau}\ cl_{\tau}\ S \neq \Phi$ and hence there exists $\Phi \neq G \in \tau$ such that $\Phi \neq G \subset cl_{\tau}\ S$. Thus $\Phi \neq G \cap A \subset A \cap cl_{\tau}\ S = cl_A S$, and $\Phi \neq int_A(G \cap A) \subset int_A cl_A S = \Phi$. This is a contradiction.

Lemma 3.2. If $Y \in \tau$, and A is nowhere dense in (X, τ), then A is nowhere dense in (Y, τ_Y).

Proof. Let $U \in \tau_Y$, then $U = Y \cap O$, $O \in \tau$. But $Y \cap O \in \tau$. then there exists $O' \in \tau$ such that $O' \subset Y \cap O$, and $A \cap O' = \Phi$. But $Y \cap O' \in \tau_Y$. Hence $A \cap (Y \cap O') = \Phi$, and A is nowhere dense in (Y, τ_Y).

Theorem 3.3. If Y is an open subspace of (X, τ). Then τ is D-equivalent with τ_Y.

Proof. By using Lemmas (3.1), (3.2).

Theorem 3.4 [4] A subset A is nowhere dense in (X,τ), iff for each $W\in\tau$, there exists $U\subset W$, $U\in\tau$ such that $U\cap A=\Phi$.

Remark. 3.1. It is easy to prove that Theorem (3.4) still valid on replacing the word open by semi-open.

For space (X,τ) we give the following properties for the class. $S(\tau)=\{\tau' : \tau'$ is a topology on X with $SO(X,\tau')=SO(X,\tau)\}$.

Theorem 3.5. If (X,τ) is a space, then τ and all members of $S(\tau)$ are D-equivalent.

Proof. By Remark (3.1), the proof is obvious.

Theorem 3.6 [4] A subset A of a space (X,τ) is dense iff $A\cap U\neq\Phi$ for each $U\in\tau-\{\Phi\}$.

Remark 3.2. It is easy to prove that Theorem (3.6) still valid on replacing the word open by semi-open.

Theorem 3.7. If (X,τ) is a space, then τ and all members of $S(\tau)$ are equivalent in the sense of Levine.

Proof. By Remark (3.2), the Proof is obvious.

Theorem 3.8. If (X,τ) is a space, then and all members of $S(\tau)$ are D-equivalent iff they are equivalent in the sense of Levine.

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Proof. By Theorems (3.5) and (3.7), the proof is obvious

Theorem 3.9. [3] If (X,τ) is an irresolvable space, $\tau(\mathcal{F})$ is a filter extension of τ by a filter $\mathcal{F}$ on X and $F \in SO(X,\tau)$, for every $F \in \mathcal{F}$, then $SO(X,\tau) = SO(X,\tau(\mathcal{F}))$.

Theorem 3.10. If (X,τ) is an irresolvable space, $\tau(\mathcal{F})$ is a filter extension of τ by a filter $\mathcal{F}$ on X and $F \in SO(X,\tau)$, for every $F \in \mathcal{F}$, then τ and $\tau(\mathcal{F})$ are D-equivalent.

Proof. By Theorem (3.9), and Remark (3.1), the proof is obvious.

Lemma 3.11. If $\tau \subset \tau'$ are two topologies on X such that they are ro-equivalent. Then the class of all nowhere dense of is contained in the class of all nowhere dense of τ'.

Proof. Let A be a nowhere dense in τ. Then $\phi = \text{int}_{\tau}\, \text{cl}_{\tau}\, A = \text{int}_{\tau'}\, \text{cl}_{\tau}\, A \supset \text{int}_{\tau'}\, \text{cl}_{\tau'} A$.

The condition that ro-equivalent is necessary as the following example showing.

Example 3.1 Let $X = \{a,b,c\}$, $\tau = \{\phi, X, \{a\}\}$ and $\tau' = \{\phi, X, \{a\}, \{b,c\}\}$, Then $RO(X,\tau) = \{\phi, X\} \neq RO(X,\tau') = \tau'$ and nowhere dense of $\tau = \{\{b\}, \{c\}, \{b,c\}, \phi\}$. But nowhere dense of $\tau' = \phi$.

Remark 3.3 Since τ_s and τ are ro-equivalent [5], then the class of all nowhere dense of τ_s is contained in the class of

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all nowhere dense of τ and thus τ_s .

Theorem 3.12. If (X,τ) is a space such that $int_{\tau} A = int_{\tau_s} A$ for each $A \in \tau^{\alpha}$. Then $\tau_s, \tau,$ and τ^{α} are D-equivalent.

Proof. By [13], we have τ and τ^{α} are D-equivalent. By Remark 3.3, If A is a nowhere dense in τ_s, then A is nowhere dense in τ

We want to prove that $\tau^{\alpha} \subset \tau_s^{\alpha}$ let $A \in \tau^{\alpha}$, then $A \subset int_{\tau} cl_{\tau} int_{\tau} A$ $= int_{\tau} cl_{\tau} int_{\tau_s} A = int_{\tau_s} cl_{\tau} int_{\tau_s} A \subset int_{\tau_s} cl_{\tau_s} int_{\tau_s} A.$

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عن التوبولوجيات المتكافئة من النوع

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في هذا البحث عرفنا ان البناء بين التوبولوجيين ح ، H على X يكونان متكافئين من النوع (أ انا كان لهما نفس العائلة من المجموعات غير الكثيفة مطلقا (nwd) اثبتنا ان التكافؤ فى مفهوم ليفين (Levine) يتطابق مع التكافؤ من النوع D انا كان للفراغين نفس العائلة من المجموعات شبه المفتوحة (semi-open) ايضا درسنا بعض الخواص والصفات للفراغات المتكافئة من النوع D . كذلك درسنا التكافؤ من النوع D بين التوبولوجى الاصلى ح وبعض من انكماشاته وتمدداته وايضا للتوبولوجيات التى لها نفس المجموعات من النوع α ، المجموعات منتظمة الفتح.