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ON ATHWART IMMERSIONS INTO EUCLIDEAN SPACE

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ABSTRACT

In this paper, athwart immersions for more than two manifolds into Euclidean space are defined. An important results for finite number of manifolds to be athwart is established. Furthermore some concluded results concerning athwart immersions into Euclidean space have been proved.

INTRODUCTION

In this work, we are concerned with a problem, namely, athwart immersions for more than two smooth, closed, connected n-manifolds into Euclidean (n + 1)-space E^{n+1}.

In [1], the same problem of athwart immersions into Euclidean space is considered for two smooth, closed, connected n-manifolds M and N into Euclidean (n + 1)-space E^{n+1}.

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Athwartness concept, as being given in [1], may be stated as follows:
Let M and N be smooth, closed, connected n-manifolds and let f and g be smooth immersions of M and N, respectively, into Euclidean (n + 1)-space E^{n+1}. The immersion f is said to be athwart to g-written $f \pitchfork g$- if and only if f(M) and g(N) have no tangent hyperplane in common.

In what circumstance is $f \pitchfork g$? was the main question which has been answered in [1] and the following theorems have been proved.

<u>Theorem (1-1).</u>

Let $f : M \rightarrow E^{n+1}$ and $g : N \rightarrow E^{n+1}$ be immersions. If f(M) has two tangent n-planes such that one meets g(N) and the other does not, then f is not athwart to g.

<u>Theorem (1-2).</u>

Let $f : M \rightarrow E^{n+1}$ and $g : N \rightarrow E^{n+1}$ be immersions such that $f(M) \cap g(N) = \phi$. Then f is not athwart to g.

<u>Theorem (1-3).</u>

Let f and g be two immersions of the unit circle S^1 in E^2. If $f \pitchfork g$, then the image of one of the immersions is inside all the loops of the other.

The most interesting result in [1] may be given as follows.

Theorem (1-4).

Let $f : M \rightarrow E^{n+1}$ and $g : N \rightarrow E^{n+1}$ be immersions such that $f \pitchfork g$. Then one of the manifolds, say M, is diffeomorphic to the n-dimensional unite sphere S^n, f is an imbedding with starshaped inside and g(N) is contained in the interior of the kernel of the inside of f.

The last theorem (1-4) depends extensively in its proof on the following theorem (1-5) by Halpern [2].

Theorem (1-5).

Let $f : M \rightarrow E^{n+1}$, dim M=n, be an immersion. If $\bigcup_{x \in M} T_x \neq E^{n+1}$ then M is diffeomorphic to S^n, f is an imbedding, the inside of f(M) is starshaped and $E^{n+1} \setminus \bigcup_{x \in M} T_x$ is the interior of the kernel of the inside of f

(T_x is the tangent space at $x \in M$).

Definition (1-1).

Let M be a smooth, closed connected n-manifolds and let f be a smooth immersion of M in an Euclidean (n+1)-space E^{n+1}. We say that f(M) is self-athwart if and only if f(M) has no tangent hyperplane touching it at more than one point.

For example, a geodesic sphere in Euclidean space is self-athwart while the 2-dimensional tours in E^3 is not self-athwart. Moreover, every strictly convex hypersurface is self-athwart and the convers is also true. [3].

In [3] an easy proof of theorem (1-1) has been given which depends mainly on the geometric imagination of geometric objects in Euclidean space and the convers of the theorem, as indicated below, has been proved.

Theorem (1-6).

Let $f : M \rightarrow E^{n+1}$ and $g : N \rightarrow E^{n+1}$ be immersions such that f is not athwart to g, then one of the immersions (or both) has two tangent n-planes such that one meets the other immersion and one does not.

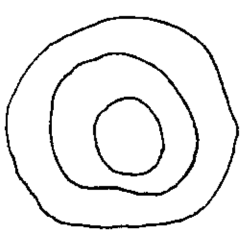
II. Main work:

Now, we discuss the above problem, athwart immersion problem, but for more than two manifolds. Firstly, we deal with three manifolds at most and then generalize the results.

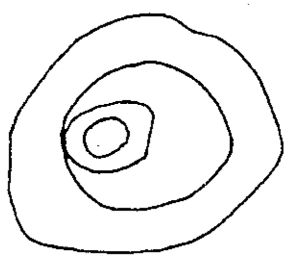
<u>Definition (2-1).</u>

Let M,N and L be smooth, closed connected n-manifolds and let f,g and h be smooth immersions of M,N and L, respectively, into Euclidean (n+1)-space E^{n+1}. We say that f,g and h are athwart immersions, denoted by $f \pitchfork g \pitchfork h$, if and only if every two of these immersions are athwart.

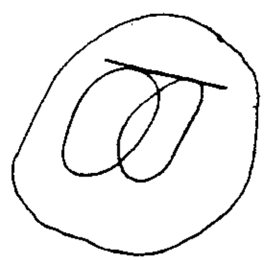
For examples, in the following figures (i) and (ii), it is clear that f,g and h are athwart immersions, while f,g and h are not athwart in figures (iii) and (iv).



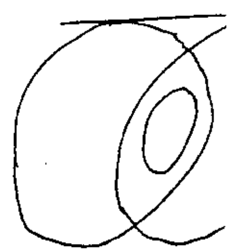
(i)



(ii)



(iii)



(iv)

Now, we prove similar theorems to the above ones.

<u>Theorem (2-1).</u>

Let $f : M \rightarrow E^{n+1}$, $g : N \rightarrow E^{n+1}$ and $h : L \rightarrow E^{n+1}$

be immersions. If any one of the immersions, say f(M), has two tangent hyperplanes such that one meets any one of the others immersions (or both), say g(N), and the other tangent hyperplane does not meet g(N) (or any one). Then f,g and h are not athwart.

Proof.

Under the above hypothesis and using theorem (1-1), we see that the immersion f is not athwart to g. According to the definition (2-1), then f,g and h are not athwart.

Theorem (2-2).

Let $f : M \rightarrow E^{n+1}$, $g: N \rightarrow E^{n+1}$ and $h: L \rightarrow E^{n+1}$ be immersions such that the intersection of any two immersions at least, say f(M) and g(N), is not empty. Then f,g and h are not athwart.

Proof.

According to the theorem (1-2), since $f(M) \cap g(N) = \phi$, then f is not athwart to g. consequently, the immersions f,g and h are not athwart.

Theorem (2-3)..

Let $f : M \to E^{n+1}$, $g : N \to E^{n+1}$ and $h : L \to E^{n+1}$ be immersions such that $f \mathbf{A} g \mathbf{A} h$. Then two of the manifolds, say M and N, are diffeomorphic to S^n, f and g are embedding with starshaped inside and h(L) is contained in the interior of the kernel of the inside of f and g.

Proof.

Since $f \mathbf{A} g \mathbf{A} h$, then we have the following cases.
(i) If we consider $f \mathbf{A} g$, then by using theorem (1-4) we see that one of the manifolds, say M, is diffeomorphic to S^n , f is an imbedding with starshaped inside and g(N) is contained in the interior of the kernel of the inside of f, i.e.,

$$g(N) \subset \ker \; f \qquad \ldots.(1)$$

(ii) If we take $g \mathbf{A} h$, then one of the manifolds, say N, is diffeomorphic to S^n , g is an imbedding with starshaped inside and h(L) is contained in the interior of the kernel of the inside of g, i.e.,

$$h(L) \subset \ker \; g, \qquad \ldots..(2)$$

(iii) If $f \mathbf{A} h$, then one of the manifolds, say M, is diffeomorphic to S^n , f is an imbedding with starshaped inside and h(L) is contained in the interior of the kernel of the inside of f. i.e. ,

$$h(L) \subset \ker f \qquad .....(3)$$

Since ker $g \subset g$, and by using (1), (2) and (3), then we have $h(L) \subset \ker g \subset \ker f$ and the theorem is proved.

<u>Theorem (2-4).</u>

Let f,g and h be immersions of the unit circle S^1 in E^2. If $f \pitchfork g \pitchfork h$, then the image of one of immersions is inside all the loops of the others.

<u>Proof.</u>

Since $f \pitchfork g \pitchfork h$ then we have the following cases:
(i) Since $f \pitchfork g$, according to the theorem (1-3), then the image of f is inside all the loops of g, or the opposite.

(ii) Since $g \pitchfork h$, then the image of g is inside all the loops of h, or the opposite.

(iii) Since $f \pitchfork h$, then the image of f is inside all the loops of h, or the opposite.

From the above three cases, we have the following:

i- If the image of f is inside all the loops of g, the image of g is inside all the loops of h and the image of f is inside all the loops of h. Or if the image of f is inside all the loops of g, the image of h is inside all the loops of g and the image of f is inside all the loops of h. Then the image of f is inside all the loops of

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g and h.

ii- If the image of f is inside all the loops of g, the image of h is inside all the loops of g and the image of h is inside all the loops of f. Or if the image of g is inside all the loops of f, the image of h is inside all the loops of g and the image of h is inside all the loops of f. Then the image of h is inside all the loops of g and f.

iii- If the image of g is inside all the loops of f, the image of g is inside all the loops of h and the image of h is inside all the loops of f. Or if the image of g is inside all the loops of f, the image of g is inside all the loops of h and the image of f is inside all the loops of h. Then the image of g is inside all the loops of f and h.

According to this discussion, we see that the image of one of the immersions is inside all the loops of the two others and the proof of the theorem is complete.

III General results.

Let $f_i : M_i \longrightarrow E^{n+1}$ be smooth immersions of smooth, closed, connected n-manifolds M_i, i= 1,2, ...,k into Euclidean (n+1)-space E^{n+1}. We say that f_i, i=1,2,...,k are athwart immersions, denoted by $f_1 \pitchfork f_2 \pitchfork \ldots \pitchfork f_k$, if and only if every two of the immersions f_i are athwart.

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Now, according to the above discussions in section (II), we have the following results:

1) If any one of the immersions f_i, say $f_j(M_j)$, $1 \leqslant j \leqslant k$, has two tangent hyperplanes such that one meets any one of the others immersions at least, say $f_r(M_r)$, $1 \leqslant r \leqslant k$, $r \neq j$, and the other tangent hyperplane does not meet any one. Then the immersions f_i are not athwart.

2) If the intersection of any two immersions is non-empty i.e. $f_r \cap f_j \neq \phi$, $1 \leqslant r$, $j \leqslant k$, $r \neq j$, then the immersions f_i are not athwart.

3) If $f_i : M_i \longrightarrow E^{n+1}$, $i = 1,2,\ldots,k$ are athwart immersions, then (k-1) of the manifolds M_i are diffeomorphic to S^n, f_i, $i = 1,2,\ldots,(k-1)$ are embeddings with starshaped inside and $f_k(M_k)$ is contained in the interior of the kernel of the inside of f_i, $i=1,2, \ldots,(k-1)$.

4) If f_i, $i=1,2,\ldots,k$ are immersions of the unit circle S^1 in E^2 and f_i are athwart immersions. Then the image of one of the immersions is inside all the loops of the others and the image of f_i is inside all the loops of f_{i+1}.

Corollary (3-1)

The set of all immersions $f_i : M_i \longrightarrow E^{n+1}$ are athwart and also self-athwart for all i if and only if they are strictly convex hyper surfaces contained in each other.

For example the set of all concentric spheres are athwart and also self-athwart.

Remark:

If we consider the set of all athwart immersions $\{f_i, i=1,2,\ldots,k\}$ such that f_i is self-athwart for all i, then we easily see that the relation of athwartness is an equivalence relation according to the following:

1- $f_i \mathcal{A} f_i$ (reflexive).

2- If $f_i \mathcal{A} f_j$ we have $f_j \mathcal{A} f_i$ (symmetric).

3- If $f_i \mathcal{A} f_j$ and $f_j \mathcal{A} f_r$ then $f_i \mathcal{A} f_r$ (transitive).

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ألاحتواءات المقترحة فى الفراغ الاقليدى

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تم فى هذا البحث دراسة الاحتواءات المعترضة (Athwart Immersions) لعدد محدود من السطوح المعممة ذات البعد ن فى الفراغ الاقليدى ذات البعد (ن+ ١) وتم استنباط نتيجة تتعلق بهذه الاحتواءات ومؤدى هذه النتيجه ان هذه السطوح المعممه تكون فى وضع احتواء معترض واحتواء معترض ذاتى اذا وفقط اذا كانت دقيقة التحدب وكل منها محتواه فى الاخر ٠