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A SPECIAL SOLUTION FOR SMALL TIME VALUES OF A DIFFUSION EQUATION

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ABSTRACT

In this paper we want to get the solution of the diffusion equation

$$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} = \frac{1}{k}\frac{\partial v}{\partial t} \quad t > 0 , \quad 0 < r < a$$

with the boundary condition

$$v(a,t) = v_0 \text{ (a constant)} \quad t \geqslant 0$$

and the initial condition

$$v(r,0) = 0 \qquad 0 \leqslant r < a$$

for small values of the time.

The principle of the method we shall use is to expand the Laplace transform of the wanted function as an asymptotic series and then invert them by term.

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A Special Solution

INTRODUCTION

In [1, pp. 46-49] the author has solved the diffusion equation

$$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} = \frac{1}{k}\frac{\partial v}{\partial t} \qquad t > 0 \, , \; o < r < a \qquad (1)$$

with the boundary condition

$$v(a,t) = f(t) \qquad t \geqslant 0 \qquad (2)$$

and the initial condition

$$v(r,0) = 0 \qquad 0 \leqslant r < a \qquad (3)$$

and we have found the solution to be

$$v(r,t) = \frac{2k}{a} \sum_{n=1}^{\infty} \frac{\alpha_n J_0(r\alpha_n)}{J_1(a\alpha_n)} \int_0^t f(u) e^{-k\alpha_n^2(t-u)} du \qquad (4)$$

where α_n are the positive zeros of $J_0(a\alpha)$.

As a special case, if

$$v(a,t) = v_0 \quad (\text{ a constant})$$

We get

$$v(r,t) = v_0\left[1 - \frac{2}{a}\sum_{n=1}^{\infty}\frac{J_0(r\alpha_n)}{{}_n J_1(a\alpha_n)}\, e^{-k\alpha_n^2 t}\right] \quad (5)$$

but as we solutions in the form of a series of terms containing exponentials such as we have just obtained are often unsuitable for use with small values of the time variable, since comptation from (5) is very laborious if Kt/a^2 is small. A new method is neede

II. Solutions for small time values.

The method we shall use is as follows:
To get the solution of the diffusion equation

$$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} = \frac{1}{k}\frac{\partial v}{\partial t} \qquad t > 0, \quad 0 < r < a \qquad (6)$$

with the boundary condition

$$v(a,t) = v_0 \text{ (a constant)} \qquad t > 0 \qquad (7)$$

and the initial condition

$$v(r,0) = 0 \quad 0 < r < a \qquad (8)$$

for small values of the time.

First we take the Laplace transform of both sides of (6) with respect to t, we get

$$\frac{d^2\bar{v}}{dr^2} + \frac{1}{r}\frac{d\bar{v}}{dr} - \frac{P}{k}\bar{v} = 0 \qquad (9)$$

The solution of this equation is

$$\bar{v} = AI_0\left\{ r\sqrt{\frac{p}{k}}\right\} + BK_0\left\{ r\sqrt{\frac{p}{k}}\right\} .$$

where I_0 and K_0 are the modified Bessel functions of zero order of the first and second kind respectively .

Since $K_0\left\{r\sqrt{\frac{p}{k}}\right\}$ tends to $-\infty$ as r tends to 0, we have $B = 0$ and so

$$\bar{v} = AI_0\left\{ r\sqrt{\frac{p}{k}}\right\} .$$

From (7), $\bar{v} = v_0/p$ where $r = a$ and we get

$$\bar{v} = \frac{v_0}{p} \frac{I_0\left\{r\sqrt{\frac{p}{k}}\right\}}{I_0\left\{a\sqrt{\frac{p}{k}}\right\}} , \qquad (10)$$

or

$$\bar{v} = \frac{v_0}{p} \frac{I_0\,(qr)}{I_0\,(qa)} , \quad \text{where} \quad q^2 = \frac{p}{k} \qquad (11)$$

Since the asymptotic expansion of $I_0(Z)$ is given by

$$I_0(z) \sim \left[\frac{e^z}{\sqrt{2\pi z}}\right]\left[1 + \frac{1}{8z} + \frac{9}{128z^2} + \cdots\cdots\right] ,$$

equation 11 a gives

$$\bar{v} = \frac{\frac{v_0}{p}\frac{e^{qr}}{\sqrt{2\pi qr}}\left[1 + \frac{1}{8qr} + \frac{9}{128q^2r^2} + \cdots\cdots\right]}{\frac{e^{qa}}{\sqrt{2\pi qa}}\left[1 + \frac{1}{8qa} + \frac{9}{128q^2a^2} + \cdots\cdots\right]}$$

$$\bar{v} = \frac{v_0}{p} e^{-q(a-r)} \sqrt{\frac{a}{r}} \left[1 + \frac{1}{8rq} + \frac{9}{128r^2 q^2} + \cdots\cdots\right]$$

$$\left[1 - \frac{1}{8aq} - \frac{9}{128a^2q^2} + \frac{1}{64a^2q^2} + \cdots\right]$$

$$= \frac{v_0}{p} e^{-q(a-r)}\sqrt{\frac{a}{r}} \left[1 + \frac{1}{8rq} + \frac{9}{128r^2q^2} - \frac{1}{8aq} - \frac{1}{64raq^2} - \right.$$

$$\left. \frac{9}{128a^2q^2} + \frac{1}{64a^2q^2} + \cdots\cdots\right]$$

$$= \sqrt{\frac{a}{r}} \frac{v_0}{p} e^{-q(a-r)} \left[1 + \frac{a-r}{8raq} + \frac{9}{128r^2q^2} - \frac{1}{64raq^2} - \right.$$

$$\left. \frac{7}{128a^2q^2} + \cdots\cdots\right]$$

$$= \sqrt{\frac{a}{r}} \frac{v_0}{p} e^{-q(a-r)} \left[1 + \frac{a-r}{8raq} + \frac{9a^2-2ra-7r^2}{128r^2a^2q^2} + \cdots\cdots\right.$$

$$\therefore \quad \bar{v} = \sqrt{\frac{a}{r}} \frac{v_0}{p} e^{-\sqrt{\frac{p}{k}}(a-r)} \left[1 + \frac{a-r}{8ra\sqrt{\frac{p}{k}}} + \frac{9a^2-7r^2-2ar}{128r^2a^2\frac{p}{k}} + \cdots\right.$$

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Using the results given in the last two entries of the table of transforms on page I, we have

$$v = v_0\sqrt{\frac{a}{r}}\left[\mathrm{erfc}\left\{\frac{a-r}{2\sqrt{kt}}\right\} + \frac{a-r}{8ar}\left\{\sqrt{kt}\ 2\mathrm{ierfc}\left\{\frac{a-r}{2\sqrt{kt}}\right\}\right\} + \ldots\right],$$

and this is useful for small values of t, provided r/a is not too small. Table for 2ierfc x is given on page 857

APPENDIX

SOME USEFUL TABLES

1) Short Table of Laplace Transforms

$$\bar{f}(p) = \int_0^\infty e^{-pt} f(t)dt.$$

$\bar{f}(p)$	$f(t)$
p^{-n-1}	$t^n/\Gamma(n+1)$, $n > -1$
$(p+\alpha)^{-1}$	$e^{-\alpha t}$
$w/(p^2+w^2)$	sin wt
$p/(p^2+w^2)$	cos wt
$(e^{-a\sqrt{p}})/p$	$\mathrm{erfc}(a/2\sqrt{t}) = (2/\sqrt{\pi})\int_{a/2\sqrt{t}}^{\infty} e^{-u^2}du$, $a > 0$
$(e^{-a\sqrt{p}})/p^{3/2}$	$2\sqrt{t}\ \mathrm{ierfc}(a/2\sqrt{t})$, where $\mathrm{i\,erfc}\ x = \int_x^\infty \mathrm{erfc}\ u\ du$, $a > 0$

2) Table for 2i erfc x

x	erfx	erfcx	2i erfc x
0	0	1.0	1.1284
0.05	0.056372	0.943628	1.0312
0.1	0.112463	0.887537	0.9396
0.15	0.167996	0.832004	0.8537
0.2	0.222703	0.777297	0.7732
0.25	0.276326	0.723674	0.6982
0.3	0.328627	0.671373	0.6284
0.35	0.379382	0.620618	0.5639
0.4	0.428392	0.571608	0.5043
0.45	0.475482	0.524518	0.4495
0.5	0.520500	0.479500	0.3993
0.55	0.563323	0.436677	0.3535
0.6	0.603856	0.396144	0.3119
0.65	0.642029	0.357971	0.2742
0.7	0.677801	0.322199	0.2402
0.75	0.711156	0.288844	0.2097
0.8	0.742101	0.257899	0.1823
0.85	0.770668	0.229332	0.1580
0.9	0.796908	0.203092	0.1364
0.95	0.820891	0.179109	0.1173
1.0	0.842701	0.157299	0.1005
1.1	0.880205	0.119795	0.0729

x	erf x	erfc x	2i erfc x
1.2	0.910314	0.089686	0.0521
1.3	0.934008	0.065992	0.0366
1.4	0.952285	0.047715	0.0253
1.5	0.966105	0.033895	0.0172
1.6	0.976348	0.023652	0.0115
1.7	0.983790	0.016210	0.0076
1.8	0.989091	0.10909	0.0049
1.9	0.992790	0.007210	0.0031
2.0	0.995322	0.004678	0.0020
2.1	0.997021	0.002979	0.0012
2.2	0.998137	0.001863	0.0007
2.3	0.998857	0.001143	0.0004
2.4	0.999311	0.000689	0.0002
2.5	0.999593	0.000407	0.0001
2.6	0.999764	0.000236	0.0001

REFERENCES

1- Abbassi, Medhat. M.: "The Solution of Partial Differential Equations by Certain Methods",Ph.D. Thesis,FACULTY OF SCIENCE, Dept. of Pure Mathematics, AIN Shams University.

2- Carslaw, H.S. and Jaeger, J.C.: Conduction of Heat in Solid, Oxford, (1947), p. 373.

3- Hartree, D.R., Mem and Proc. Manchester lit. and Phil. Soc., Vol. 80, (1935). p. 85.

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الحل الخاص لمعادلة الانتشار فى حالة القيم الصغيرة للزمن

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فى هذا البحث نرغب فى ايجاد حل معادلة الانتشار

$$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} = \frac{1}{k}\frac{\partial v}{\partial t} \quad t > o \, , \, o < r < a \, ,$$

تحت الشرط الحدى

$$v(a,t) = v_o \quad (\text{a constant}) \quad t > o$$

والشرط الابتدائى ،

$$v(r,o)=o \quad , \quad o \leq r < a$$

لقيم t الصغيرة ٠

الطريقة الاساسية المستخدمة هى فك تحويل لا بلاس للدالة المطلوبة على شكل متسلسلة تقاربية ثم ايجاد الحدود ٠