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A QUADRATIC OPTIMAL SYNTHESIS PROBLEM WITH FAILURE CONTROL

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ABSTRACT

A quadratic optimal control problem with control component failure is considered in this work. The concept of recoverability is presented and applied to a constant linear quadratic control system. A second order control problem being analyzed with the objective to minimize a certain regulized cost functional is given. An optimal synthesis is determined for a control system without and with a control component failure. Both the singular and nonsingular optimal controls are obtained in a simple structure involving linear state and switching functions.

INTRODUCTION

The optimal control problem for linear system with a saturable scalar control has been extensively studied for various cost functionals, [2,3,5] ,. In the case of time optimal and fuel optimal control, explicit analytical solutions have been obtained for many systems of low order, [3,6],.

In control problems it can happen that some components of the control fails for an indefinite period of time before the objective is accomplished. When the system works to avoid this fault, it will tend to accomplishe the objective with the remaining components of control which are operating normally.

In this work an optimal control system having a certain quadratic cost functional with a failure control, is studied. The cost integrand will be modified by introducing a suitably chosen non-quadratic state penalty term. This modification regularizes the problem in order to obtain explicit synthesis forms for both singular and nonsingular optimal controls. The concept of recoverability, [1], is developed and applied to the control system under consideration . The optimal problem without and with failure components will be considerd. Both the singular and nonsingular optimal controls are obtained in a simple structure involving linear state and switching functions. The analysis will be applied to the two dimensional control system , in its general form, with and without component failure. Two illustrative examples will be given to explain the feasibility of minimizing the regulized cost functional under the action of the active components of the control vector.

The following assumptions are necessary for the forthcoming analysis,

1- In the case of two dimensional control system, only one control parameter fails at time τ less than the final time t_f for an indefinite period of time and no further failures occur.
2- The elapsed time between the failure and the formulation of a new control synthesis is zero.
3- When the component of control fails, the available cost and the system states are observable.

2- LINEAR CONTROL SYSTEM

Consider the completely controllable and observable linear time invariant multivariable control system:

$$\dot{X} = A X + B U \qquad (2\text{-}1)$$

where the dimensions of the state vector X(t) and the control vector U(t) are (n . 1) and (m . 1), respectively. The matrix B has a full rank . It is well known that for a finite time regulation, the control function U*(t) which minimize the cost functional

$$J(U)=\tfrac{1}{2} < X^T(t_f),F\ X(t) > + \int_{t_0}^{t_f}[<X,\ QX>+ < U,RU>]dt \qquad (2\text{-}2)$$

is given by

$$U^*(t) = -R^{-1}B^TK(t) X(t) \qquad (2\text{-}3)$$

where (i) the matrix Q is of the form M^TM with the pair (A,M) being completely observable,

(ii) the matrix F is positive semidefinite, and

(iii) the matrix F is positive semidefinite, and

(iv) the matrix K(t) is a unique positive definite solution to the matrix Riccati equation

$$A^TK + KA - KBR^{-1} B^TK + Q = 0 \qquad (2\text{-}4)$$

satisfying the boundary condition

$$K(t_f) = F \qquad (2\text{-}5)$$

The state of the optimal system is then the solution of the linear differential system

$$\dot{X} = [A - BR^{-1} B^TK] X (t) \qquad (2\text{-}6)$$

The following two lemmas, which can be proved, [6], gives the necessary conditions for the existence and uniqueness of the optimal control.

Lemma 1.

The control vector given by (2-3) yields to a local minimum for the cost functional (2-3).

Lemma 2.

If an optimal exists, then it is unique and is given-by (2-3).

In the following section a modified cost functional will be introduced by adding a suitably chosen non-quadratic state penalty term in the cost integrand (2-2). This modification is necessary to avoid solving Riccati differential equation (2-5) and to obtain an explicit synthesis forms for both sigular and nonsigular optimal controls. The concept of recoverability for a linear quadratic will be developed and applied to the control system under consideration.

3- RECOVERABILITY OF A LINEAR QUADRATIC SYSTEM.

Consider the linear control system (2-1), where the control restraint set Ω is taken to be the unit cube in R^m , and the matrix B has a full rank . Let r-components of the control vector U(t) fail at time τ , $\tau < t_f$. The modified system is assumed to be

$$\dot{X} = A X + \bar{B} V (t) \qquad (3-1)$$

where $V(t) \in R^{m-r}$.

Without loss of generality, the cost functional (2-2) will be modified by assuming that the matrix R equals zero and the integrand of the cost functional takes the form

$$\tfrac{1}{2} < X , Q X > + \sum_{i=1}^{m-r} | < C_j(t), X(t) >| \qquad (3-2)$$

where the m-r dimensional vectors $C_j \in R^n$ are continuous functions to be determined. The corresponding Hamilitonian can be written as

$$H(t,X,U,\eta)=-\tfrac{1}{2}\langle X,QX\rangle - \sum_{j=1}^{m-r}\left|\langle C_j,X\rangle\right| + \langle \eta,AX\rangle+\langle \eta , \bar{B}V\rangle \tag{3-3}$$

In accordance with Pontryagin maximum principle, the control which minimizes the cost functional has a cost integrand of the form (3-2), and must satisfy the following relation

$$H(t,X(t),V(t),\eta(t))= \max_{V\in\Omega} H(t,X(t), V,\eta(t)) \tag{3-4}$$

for some $t\in[t_0, t_f]$. The adjoint vector (.) is the solution of

$$\dot{\eta}(t) = -A^T\eta(t)+QX+ \sum_{j=1}^{m-r} C_j \ \mathrm{sgn}\langle C_j,X\rangle \tag{3-5}$$

with the boundary condition

$$\eta(t_f) = -F\, X\,(t_f) \ . \tag{3-6}$$

The nonsigular optimal control which minimize the Hamilitonian (2-3), is given by

$$v_j^*(t) = \mathrm{sgn}\langle b_j(t) , \eta(t)\rangle \tag{3-7}$$

while the singular control will arise on a singular interval

$T_s \subset [t_0 , t_f]$ if

$$< b_j(t0), \eta(t)>=\theta , \quad j=1,2,.....,m-r. \qquad (3-8)$$

Assume that the adjoint vector $\eta(t)$ is related to the state vector X(t) via the symmetric, linear transformation

$$\eta(t) = K(t)\ X(t) \qquad (3-9)$$

where K(t) is a symmetric matrix function, to be determine, satisfaying the boundary condition (2-5). The nonsigular control (3-7) can be expressed as follows

$$V_j^* (t,x) = \text{sgn} < K(t)\ b_j\ (t),\ x> \qquad (3-10)$$

Substituting (3-9) in (3-5) yields

$$\dot{K}\ (t) + K(t)\ A+A\ K(t)-Q\ X(t) + \sum_{j=1}^{m-r} K(t)b_j\ \text{sgn} <K(t)b_j,\ X>-$$

$$- C_j\ \text{sgn} < C_j,\ X > \ =0 \qquad (3-11)$$

If C_j is defined to be

$$C_j = K(t)\ b_j\ (t) \qquad (3-12)$$

then (3-11) will be reduced to the following Lyapunov matrix differential equation

$$\dot{K}(t) + K(t)\ A(t) + A^T(t)\ K(t)-Q(t)=0 \qquad (3-13)$$

which can be solved by using the boundary condition (2-5).

The cost functional (3-2) is then takes the form

$$J(V)=\tfrac{1}{2}\langle X(t_f),F X\rangle + \tfrac{1}{2}\int_{t_o}^{t_f} \langle X,QX\rangle+\sum_{j=1}^{m-r} |\langle K(t)b_j(t),X(t)\rangle|]dt \qquad (3\text{-}14)$$

Concerning the singular interval T_s , we assume that the optimal trajectory may lie on one or more of the singular sets

$$Y_{js}(t) = \left\{ X\in R^n \ \langle K(t)b_j(t) , X\rangle 0 \right\} \qquad (3\text{-}15)$$

To simplify the forthcomming analysis we assume that only of the control components is singular while all the others are of nonsingular nature.

On the singular interval T_s , we differentiate relation (3-8).

Using equations (3-1) and (3-13), the singular control can be determined by

$$v_{sp} = \xi_{pp}^{-1} \quad \psi_p(t,X) \qquad (3\text{-}16)$$

where

$$\xi_{pp} = \langle K(t)b_p(t) , b_p(t)\rangle \quad , \qquad (3\text{-}17)$$

$$\psi_p(t,X) = \langle [Q(t)-K(t)A(t)]b_p(t),X\rangle+\langle K(t)\tilde{b}_p(t),X\rangle+ + \sum_{j\neq p}^{m-r} \langle K(t)b_p(t),b_j(t)\rangle \mathrm{sgn} \langle K(t)b_j(t),X\rangle \quad (3\text{-}18)$$

In view of the control constrain $\ddot{v}_p(t) < 1$, it follows that the admissible family of such singular arcs constitutes a subset Y_{ps} of Y_{js} with the characterization

$$Y_{ps} = \left\{ (t,X) \in [t_0,\ t_f] \times R^n \mid X \in Y_p\ ,\ X \notin Y_j(t)\ \forall\, j \neq p, |\psi_p(t,X)| < |\varrho_{pp}(t)| \right\} \tag{3-19}$$

The above analysis proves the following theorem

Theorem 1.

For the linear control system under consideration, if r components of the control fail at time τ, $\tau < t_f$, then the remaining components which satisfy system (3-1) and minimizing the cost functional (3-14) are optimal . This optimal control decomposed to a nonsingular control given by (3-10) and a singular one given by (3-16).

In the next section the study will be restrict to the case of two dimensional control system.

4-TWO DIMENSIONAL SYSTEM WITHOUT COMPONENT FAILURE.

In this case control system (2-1) can be written as

$$\begin{aligned} \dot{X}_1 &= a_{11}x_1 + a_{12}x_2 + b_{11}u_1 + b_{12}u_2 \\ \dot{X}_2 &= a_{21}x_1 + a_{22}x_2 + b_{21}u_1 + b_{22}u_2 \end{aligned} \tag{4-1}$$

where $b_{11}b_{22} - b_{21}b_{12} \neq 0$. Assume that no component of the control vector is failed . To simplify the calculation, consider the following two linear transformations,

a) There existes a nonsingular matrix M such that

$$A_1 = M^{-1} AM \qquad (4\text{-}2)$$

is one of the Jourdan canonical forms.

b) Assume that

$$U_1 = M^{-1}B\ U \quad , \quad U_1 = (u_{11} \quad u_{12})^T \qquad (4\text{-}3)$$

Under these two transformations , the control system (4-1) becomes

$$\dot{x}_1 = \lambda_1 x_1 + u_{11} \qquad (4\text{-}4)$$

$$\dot{x}_2 = \lambda_2 x_2 + u_{12} \qquad (4\text{-}4)$$

where λ_1 and λ_2 are the two real distinct eigenvalues of the system matrix. Let the matrix Q be a diagonal one with two diagonal elements q_{11} and q_{22} . The cost functional (3-2) takes the form

$$J(U_1) = \int_{t_0}^{t_f} (\tfrac{1}{2}q_{11}x_1^2 + q_{22}\, x_2^2) + (\,|\langle c_1, X(t)\rangle| + |\langle c_2 , X(t)\rangle|\,))\, dt \qquad (4\text{-}5)$$

where c_1 and c_2 are two 2-dimensional constant vectors to be determined, Since the system matrix A_1 is symmetric and constant, then the matrix Lyapunov equation (3-13) will be reduced to

$$K\,A + A\,K - Q = 0 \qquad (4\text{-}6)$$

or equivalently

$$\begin{aligned} 2\lambda_1 K_{11} - q_{11} &= 0 \\ 2\lambda_2 K_{22} - q_{22} &= 0 \\ (\lambda_1 + \lambda_2)\,K_{12} &= 0 \end{aligned} \qquad (4\text{-}7)$$

The symmetric solution matrix K is

$$K = \begin{bmatrix} q_{11}/2\lambda_1 & 0 \\ 0 & q_{22}/2\lambda_2 \end{bmatrix} \qquad (4\text{-}8)$$

where $\lambda_1 , \lambda_2 \neq 0$ and $\lambda_1 + \lambda_2 \neq 0$.

The nonsingular controls may be expressed as follows

$$\begin{aligned} u^{*}_{11}(X) &= \operatorname{sgn}(q_{11}/2\lambda_1 x_1) \;;\; x_1 \neq 0 \\ u^{*}_{12}(X) &= \operatorname{sgn}(q_{22}/2\lambda_2 x_2) \;;\; x_2 = 0 \end{aligned} \qquad (4\text{-}9)$$

on the sigular interval T_s, the optimal trajectoty X(t) lies on one or more of the following singular sets

$$Y_{1s} = \left\{ (x_1, x_2) \in R^2 \mid x_1 = 0 \,,\; x_2 \neq 0 \,,\; |x_2| \leqslant 1 \right\} \qquad (4\text{-}10)$$

$$Y_{2s} = \left\{ (x_1, x_2) \in R^2 \mid x_2 = 0, \quad x_1 \neq 0 , \quad |x_1| \leqslant 1 \right\} \quad (4-11)$$

Assume that only the first control is singular, then the corresponding $\Psi_1(t)$ and $\varrho_{11}(t)$ can be written as

$$\Psi_1(t) = -k_{12} \lambda_1 x_2 , \qquad |x_2| \leqslant 1 \quad (4-12)$$

$$\varrho_{11}(t) = q_{11} / 2\lambda_1 \quad (4-13)$$

Then the first singular control is

$$u^*_{11s} = |2\lambda_1 / q_{11}| K_{12} \lambda_1 x_2 \quad (4-14)$$

This singular control will be equal to zero on the singular set Y_{1s}. Similarly on Y_{2s} the second singular control will also equal zero . The optimal control can be written as follows

$$u^*_{11}(X) = \begin{cases} \text{sgn } q_{11}/2\lambda_1 \ x_1 & (x_1, x_2) \notin Y_{1s} \\ 0 & (x_1, x_2) \in Y_{1s} \end{cases}$$

$$u^*_{12}(X) = \begin{cases} \text{sgn } q_{22}/2\lambda_2 \ x_2 & (x_1, x_2) \notin Y_{2s} \\ 0 & (x_1, X_2) \in Y_{2s} \end{cases} \quad (4-15)$$

These two external control parameters minimizing the following cost functional are :

$$J(U_1)= \int_{t_0}^{t_f} (\tfrac{1}{2}(q_{11}x_1^2 + q_{22}\, x_2^2)+ (q_{11}/\, 2\lambda_1\, x_1+q_{22}/2\lambda_2 x_2\,))dt \tag{4-16}$$

In the following example, the case of two complex eigenvalues will be considerd .

Example 1.

Consider the control system

$$\dot{x}_1 = x_2 + u_1$$

$$\dot{x}_2 = -2x_1 - 2x_2 + u_2 \tag{4-17}$$

with a cost functional of the form

$$J(U) = \int_{t_0}^{t_f} (2x_1^2 + x_2^2 + |\langle c_1, X\rangle| + \langle c_2, X \rangle)\, dt$$

The corresponding matrix Lyapunov equation (3-13) will be reduced to take the same form as (4-6) with solution as

$K_{11} = -4$, $k_{12} = K_{21} = K_{22} = -1$.

consequentely the two constant vectors c_1 and c_2 are to be

$c_1= (-4 \quad -1)^T$ and $c_2= (-1 \quad -1)^T$

The corresponding nonsingular controls are

$u_1^*\,(x_1,x_2) = -\ \text{sgn}\ (4x_1 + x_2)$, and

$u_2^*(x_1, x_2) - \text{sgn}(x_1 + x_2)$.

On the interval T_s, X(t) lies on one more of the following singular sets

$$Y_{1s} = \{(x_1, x_2) \in R^2 / 4x_1 + x_2 = 0 \quad , \quad x_1 + x_2 \neq 0\},$$

$$Y_{2s} = \{(x_1, x_2) \in R^2 / x_1 + x_2 = 0 \quad , \quad 4x_1 + x_2 \neq 0\},$$

$$Y_{1,2s} = \{(x_1, x_2) \in R^2 / 4x_1 + x_2 = 0 \quad , \quad x_1 + x_2 = 0\}.$$

On T_s assume that only the first control is singular, then it is given by

$$u_{1s}^*(x_1, x_2) = -1/4\,(10x_1 - 3 \quad \text{sgn}\, x_1)$$

Similarly if u_2 is singular then

$$u_{2s}^*(x_1, x_2) = x_1 + 3 \quad \text{sgn}\, x_1 \quad ; \quad x_1 = -x_2 .$$

The required optimal control is given by

$$u_1^*(x_1, x_2) = \begin{cases} -\ \text{sgn}\ (4x_1 + x_2); & (x_1 + x_2) \notin Y_{1s} \\ -\ 1/4(10x_1 - 3\ \ \text{sgn}\ \ x_1) \ ; & (x_1, x_2) \in Y_{1s} \end{cases}$$

$$u_2^*(x_1, x_2) = \begin{cases} -\ \ \text{sgn}\ (x_1 + x_2)\ ; & (x_1, x_2) \notin Y_{2s} \\ x_1 + 3\ \ \text{sgn}\ x_1 \ ; \ x_1 = -x_2 & ; (x_1, x_2) \in Y_{2s} \end{cases}$$

and minimizes the following cost functional

$$J(U) = \int_{t_0}^{t_f} (2x_1^2 + x_2^2 + |-4x_1 + x_2| + |-x_1 + x_2|)\, dt$$

Considering the above analysis mentioned in section 3, the control problem with a failure component will be given throught the next example.

Example 2.

Let the component u_1 of the control system described by

$$\dot{x}_1 = x_2 \quad ; \quad \bar{x}_2 = -x_1 + u_2$$

fails at time τ befor the objective is accomplished. In this case the corresponding cost functional takes the following from

$$J(U)= \int_{t_0}^{t_f} (2x_1^2 + x_2^2 + |< c_1 , X >|)\, dt$$

Here the nonsingular control will be given by

$$U_2^* (x_1, x_2) = - \quad \text{sgn}\ (x_2) \quad ; \quad x_2 \neq 0$$

while on T_s the optimal trajectory lie on the set

$$Y_{2s} = \left\{ (x_1 , x_2) \in R^2 \,|\, x_2 = 0; \quad x_1 \neq 0 \quad ; |x_1| \leq 1 \right\}$$

and the singular control is given by

$$u_{2s}^* \ (x_1 , x_2) = x_1 .$$

This proves that the optimal control defined by

$$u_2^*(x_1,x_2)\begin{cases} \operatorname{sgn} x_2 & \text{if } (x_1, x_2) \notin Y_{2s} \\ x_1 & \text{if } (x_1, x_2) \in Y_{2s} \end{cases}$$

will minimize the following cost functional

$$J(U) = \int_{t_0}^{t_f} (2x_1^2 + x_2^2 - x_2)\,dt$$

If failure occur on u_2 , then the problem can be solved in a similar way .

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مشكلة السينزيز المثالى لنظام تربيعى حيث يفشل احد عوامل التحكم

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ان مشكلة التحكم الامثل لنظام تربيعى حيث احد عوامل التحكم يفشل فى السيطرة لفتره محدوده اخذت فى الاعتبار فى هذا البحث ٠ ان مبدأ الاستعاده اخذ فى الاعتبار وطبق على نظام خطى تربيعى ثابت ٠

لقد اعطينا تحليلا لنظام من الرتبه الثانية دالته التكلفية منتظمة ٠ تم تحديد دالة السينزيز المثالى للنظام فى حالتين ، اولا حيث لا يفشل اى عامل تحكم فى السيطرة على النظام ، ثانيا حيث يفشل احد عوامل التحكم فى السيطرة ٠ وفى كلا الحالتين فان عامل التحكم المثالى الشاذ وغير الشاذ تم الحصول عليه كدالة فى كل من الحالة الخطية ودوال الأبدال ٠