THE 3- PLANE AND THE LIGHT CONE

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ABSTRACT

In this paper we show that the 3-plane passing through the origin in a space-time will intersect the light cone in two perpendicular 2-planes.

1- The Principal Planes:

In this section we will give a sketch of how the principal planes can be obtained in order to be able to discuss the way in which a 3-plane intersect the light cone in space time.

A principal plane is a diametral plane which is at right angles to the chords which it bisects, Now if the axes are rectangular, the diametral plane whose equation is

$$\ell \frac{\partial F}{\partial x} + m \frac{\partial F}{\partial y} + n \frac{\partial F}{\partial z} = 0, \text{ where}$$

$$F(x,y,z) \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + d = 0$$

or $x(a\ell + hm + gn) + y(h\ell + bm + fn) + z(g\ell + fm + cn) = 0,$

is at right angles to the line $\dfrac{x}{\ell} = \dfrac{y}{m} = \dfrac{z}{n}$, if

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$$\frac{a\ell + hm + gn}{\ell} = \frac{h\ell + bm + fn}{m} = \frac{g\ell + fm + cn}{n}$$

If each of these ratios is equal to λ, then

$$(a - \lambda)\ell + hm + gn = 0,$$

$$h\ell + (b - \lambda)m + fn = 0, \qquad (i)$$

$$g\ell + fm + (c - \lambda)n = 0$$

Therefore , λ is a root of the equation:

$$\begin{vmatrix} a - \lambda & h & g \\ h & b - \lambda & f \\ g & f & c - \lambda \end{vmatrix} = 0$$

or equivalently,

$$\lambda^3 - \lambda^2(a+b+c) + \lambda(bc+ca+ab-h^2-g^2-f^2) - D = 0 \qquad (ii)$$

where $D \equiv \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$

Equation (ii) is called the discriminating cubic. It gives three values of λ, to each of which corresponds a set of values for (ℓ ,m,n) and by substituting these sets in the equation $\ell \frac{\partial F}{\partial x} + m \frac{\partial F}{\partial y} + n \frac{\partial F}{\partial z} = 0$, which by means of

the relations (i) reduces to $\lambda(\ell x + my + nz) = 0$, we obtain

the equations of three principal planes [1].

2- The Main Result:

A 3-plane in space time has the equation $A_r x_r + B = 0$,

$r = 0,1,2,3$. This equation will reduce to

$$A_r x_r = 0, \quad r = 0,1,2,3,4, \tag{1}$$

if the 3-plane is passing through the origin. In this case

it will have a unique orthogonal line through the origin

with equations $x_r = A_r u$, where u is a parameter.

Now consider the equation of the light cone:

$$\sum_{i=1}^{3} x_i^2 - x_4^2 = 0 \tag{2}$$

and rewrite equation (1) as follows:

$$\sum_{i=1}^{3} B_i x_i + x_4 = 0 , \quad B_i = A_i / A_4 \tag{1'}$$

From (1)' and (2) we have

$$\left(\sum_{i=1}^{3} B_i x_i \right)^2 = (-x_4)^2 = x_4^2 = \sum_{i=1}^{3} x_i^2$$

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$$\therefore \sum_{i=1}^{3} B_i^2 x_i^2 + 2 \sum_{\substack{i,j=1 \\ i \neq j}}^{3} B_i B_j x_i x_j = \sum_{i=1}^{3} x_i^2$$

or

$$\sum_{i=1}^{3} C_i^2 x_i^2 + 2 \sum_{\substack{i,j=1 \\ i \neq j}}^{3} B_i B_j x_i x_j = 0, \quad C_i^2 = B_i^2 - 1 \tag{3}$$

The discriminating cubic for equation (3) is:

$$\lambda^3 - \lambda^2 (C_1^2 + C_2^2 + C_3^2) + \lambda(C_2^2 C_3^2 + C_1^2 C_2^2 - B_2^2 B_3^2 - B_1 B_3^2 - B_1^2 B_2^2) - D = 0,$$

where: $D \equiv \begin{vmatrix} C_1^2 & B_1 B_2 & B_1 B_3 \\ B_2 B_1 & C_2^2 & B_2 B_3 \\ B_1 B_3 & B_2 B_3 & C_3^2 \end{vmatrix} = C_1^2 + C_2^2 + C_3^2 + 2 \neq 0$

Thus: $\lambda^3 - A \lambda^2 - (2A + 3)\lambda - (A + 2) = 0$, $A = C_1^2 + C_2^2 + C_3^2$,

and so, $(\lambda + 1)^2 [\lambda - (A + 2)] = 0$. It follows that $\lambda = -1, -1, A+2$.

Consider first $\lambda = -1$. In this case the set of equations (i) may reduce to the single equation:

$$B_1 \ell_1 + B_2 m_1 + B_3 n_1 = 0, \quad i = 1,2,3 \tag{4}$$

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If we consider $\lambda = A+2$, the set of equations (i) may take the form:

$$(B_2^2 + B_3^2)\ell_3 + B_1B_2 m_3 + B_1B_3 n_3 = 0,$$

$$B_1B_2\ell_3 - (B_1^2 + B_3^2) m_3 + B_2B_3 n_3 = 0, \qquad (5)$$

$$B_1B_3\ell_3 + B_2B_3 m_3 - (B_1^2 + B_2^2) n_3 = 0$$

Dividing the first equation of (5) by B_1 and the second by B_2 then substracting, we have:

$$\frac{\ell_3}{B_1} = \frac{m_3}{B_2}$$

Again from the second and third equations of (6), we get

$$\frac{m_3}{B_2} = \frac{n_3}{B_3}$$

Thus,

$$\frac{\ell_3}{B_1} = \frac{m_3}{B_2} = \frac{n_3}{B_3} \qquad (6)$$

From the above results we find that the single equation (4) corresponding to $\lambda_1 = \lambda_2 = -1$, is the condition that the directions given by (ℓ_1, m_1, n_1) and (ℓ_3, m_3, n_3) should be at right angles. The principal planes corresponding to the directions (ℓ_1, m_1, n_1) and (ℓ_3, m_3, n_3) are respectively : $\ell_1 x_1 + m_1 x_2 + n_1 x_3 = 0$ and $\ell_3 x_1 +$

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$m_3 x_2 + n_3 x_3 = 0$, or equivalently : $\ell_1 x_1 + m_1 x_2 + n_1 x_3 = 0$ and $B_1 x_1 + B_2 x_2 + B_3 x_3 = 0$, where $B_1 \ell_1 + B_2 m_1 + B_3 n_1 = 0$. It follows that the 3-plane which pass through the origin will intersect the light cone in two perpendicular planes.

REFERENCE

1- T.J. Willmore, An Introduction to Differential Geometry, Oxford University Press, London, 2nd. Impression, 1982.

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