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NEW PROPERTIES OF PREOPENNESS

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ABSTRACT

This paper is devoted to studying several properties of a type of near openness which is called preopen sets, also new results of precontinuity are established and connections between this type with corresponding ones are studied. Some existing results are improved.

1. INTRODUCTION

In 1963, Levine [7] introduced the notion of semi-open set and semi-continuity, but α - sets has been defined by Njastad [11] in 1965. Locally closed sets was given in 1966 by Bourbaki [4]. The concept of δ -open sets was defined by Velicko [15] in 1968. In 1980, Mashhour et al. [8] introduced the concept of a preopen sets which is independent from semi-openness. In 1983, Abd El-Monsef et al. [1] introduced the notion of β-openness. In 1986, Tong [14] gave the notion of A-set.

Delta J.Sci. 12 (2)1988

Here X,Y and Z are topological spaces without any separation properties and whenever such ones are needed they will be assumed. Let S be a subset of X, the closure (resp. interior) of S will be denoted by $\bar{S}$ (resp. S°). S is called semi-open [7] (resp. α-set [11], locally closed [4], preopen [8], β-open [1], A-set [13]) if $S \subset S^{\circ -}$ (resp. $S \subset S^{\circ - \circ}$, $S = U \cap F$ where U is open and F is closed, $S \subset S^{-\circ}$, $S \subset S^{-\circ -}$, $S = U \cap F$ where U is open and F is regular closed). The complement of each of the previous types will be called a corresponding closed. S is called δ-open [15] if for each point $x \in S$ there exists a regular open set G containing x such that $G \subset S$. The complement of a δ-open set is δ-closed [15]. The collection of all semi-open (resp. α-sets, preopen, β-open, regular-open, regular-closed) sets of X will be denoted by SO(X) (resp. αO(X), PO(X), βO(X), RO(X), RC(X)). A mapping $f;X \longrightarrow Y$ is said to be semi-continuous [7] (resp. α-continuous [10]. LC-continuous [6], precontinuous. [8], β-continuous [1], A-continuous [14]) if the inverse image of each open set in Y is semi-open (resp. α-set, locally closed, preopen, β-open, A-set) in X, also f is almost continuous in the sense of Singal [13] (a.c.s) (resp.irresolute [5], M-precontinuous [9], β-irresolute [2]) if the inverse image of each regular-open (resp. semi-open,preopen, β-open) set in Y is open (resp. semi-open, preopen, β-open) in X. But f is semi-open [3] (resp. semi-closed [12], β-open

[1], preopen [8], almost open in the sense of Singal [13] (a.o.s.), M-preopen [9], M-preclosed [9]) if the image of each open (resp. closed, open, α-open, regular open, preopen, preclosed) set in X is semi-open (resp. semi-closed, β-open, preopen, open, preopen, preclosed) set in Y.

2. PREOPEN SETS

Theorem 2.1. The intersection of a preopen set and an α-set is preopen.

Proof Let $A \in PO(X)$ and $B \in \alpha O(X)$, then $A \subset A^{-\circ}$, $B \subset B^{\circ-\circ}$ so, $A \cap B \subset A^{-\circ} \cap B^{\circ-\circ} \subset (A^{-\circ} \cap B^{\circ-})^{\circ} \subset (A^{-} \cap B^{\circ})^{-\circ} \subset (A \cap B)^{=\circ} = (A \cap B)^{-\circ}$, hence the result.

Corollary 2.1. The union of a preclosed set and, a α-closed set is preclosed.

The following theorem is an improvement of a corresponding result in [15]

Theorem 2.2. A subset P of X is δ-closed iff for each $x \notin P$, there exists a preopen set W with $x \in W$ such that $W^{-\circ} \cap P = \emptyset$.

Proof. Let $P \subset X$ be δ-closed, then X-P is δ-open i.e., for each $x \in X-P$, there exists a regular-open set $W^{-\circ} \subset X$ such that $x \in W^{-\circ} \subset X - P$. Hence $W^{-\circ} \cap P = \emptyset$.

Delta J.Sci. 12 (2)1988
New Properties of Preopenness

Conversely, let $x \in P$, there esists a pre-open set $W \subset X$ containing x such that $W^{-\circ} \cap P = \emptyset$. Since $W^{-\circ}$ is regular open containing x, then $x \in W^{-\circ} \subset X-P$ this means that X-P is δ-open and therefore P is δ-closed.

Definition 2.1[16]. A space X is said to be extremely disconnected if the closure of every open set in it is open. By using Definition (2.1), the following result having obviously proof.

Theorem 2.3. In an extremely disconnected space, the following statements hold :-

(i) Each β-open set is preopen.
(ii) Each β-closed set is preclosed.
(iii) Each semi-open set is α-set.
(iv) Each semi-closed set is α-closed.

3. PRECONTINUITY

Theorem 3.1. A mapping $f : X \longrightarrow Y$ is precontinuous iff $f(A^{-}) \subset [f(A)]^{-}$ for every $A \in SO(X)$.

Proof. Necessity, let f be precontinuous and $A \in SO(X)$, then $A \subset A^{\circ -}$ and $A^{-} \subset A^{\circ -}$. Hence $f(A^{-}) \subset f(A^{\circ -})$. By precontinuity of f we have $f(A^{\circ -}) \subset [f(A)]^{-}$ (Theorem (1) [8]) hence $f(A^{-}) \subset [f(A)]^{-}$.

Sufficiency, let $f(A^{-}) \subset [f(A)]^{-}$ for each $A \in SO(X)$.

Since $A^{\circ -} \subset A^{-}$, $f(A^{\circ -}) \subset f(A^{-}) \subset [f(A)]^{-}$. Then by Theorem (1) [8], we get directly that f is precontinuous.

Theorem 3.2. If $f : X \longrightarrow Y$ is a.c.s. and β-open, then f is precontinuous.

Proof. Let $V \subset Y$ be open and since $V^{-\circ}$ is regular open in Y. Then $f^{-1}(V^{-\circ}) = [f^{-1}(V^{-\circ})]^{\circ}$ but $V \subset V^{-\circ} = V^{\circ -\circ}$. Thus $f^{-1}(V) \subset [f^{-1}(V^{\circ -\circ})]^{\circ}$ but by β-openness $f^{-1}(V) \subset [f^{-1}(V)]^{-\circ}$ (see[1]). Hence $f^{-1}(V) \in PO(X)$ and this complete the proof.

Theorem 3.3. A β-continuous mapping from an extremely disconnected space into any space is precontinuous.

Proof. Let $f : X \longrightarrow Y$ be β-continuous and $V \subset Y$ is open, then $f^{-1}(V) \in \beta O(X)$. By Theorem (2.3) $f^{-1}(V) \in PO(X)$. Hence the result.

Theorem 3.4. If $f : X \longrightarrow Y$ is precontinuous and open then :

(i) f is M-precontinuous.

(ii) The inverse image of each semi-open (resp. semi-closed) set is β-open (resp. β-closed).

Proof. (i) Let $W \in PO(Y)$, then $W \subset W^{-\circ}$. Since $W^{-\circ}$ is open and f is precontinuous, then $f^{-1}(W) \subset f^{-1}(W^{-\circ}) \subset [f^{-1}(W^{-\circ})]^{-\circ} \subset [f^{-1}(W^{-})]^{-\circ} \subset [f^{-1}(W)]^{-\circ}$, hence the result.
(ii) If $V \in SO(Y)$, then $V \subset V^{\circ -}$ and so $f^{-1}(V) \subset f^{-1}(V^{\circ -}) \subset [f^{-1}(V^{\circ})]^{-}$ and by precontinuity of f we have $f^{-1}(V) \subset$

$[f^{-1}(V^\circ)]^{-\circ-} \subset [(f^{-1}(V)]^{-\circ-}$. Hence $f^{-1}(V) \in \beta O(X)$ and the other part of this statment follows directly.

Corollary 3.1. (i) If X is extremely disconnected then the second statement of the previous theorem will be "The inverse image of each semi-open (resp. semi-closed) set is preopen (resp. preclosed)".
(ii) If each of X and Y are extremely disconnected, then the inverse image of each α-set (resp. α-closed set) is preopen (resp. preclosed).

Theorem 3.5. For any two mappings $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$, the following statements hold :-
(i) If f is surjective precontinuous and open, then gof is precontinuous if one of the following verifies :
1) If g is precontinuous.
2) If Y is extremely disconnected and g is β-continuous.
(ii) If gof is precontinuous and g is continuous, then the surjective mapping f is precontinuous.
(iii) gof is precontinuous if f is precontinuous, g is A-continuous and Y is extremely disconnected.
Proof. Obvious.

4. PREOPEN AND PRECLOSED MAPPINGS

Lemma 4.1.[10]. Let $f : X \longrightarrow Y$ be α-continuous, then $f(A^-) \subset [f(A)]^-$ for each $A \in PO(X)$.

Theorem 4.1. Preopenness is preserved under preopen and α-continuous mapping.

Proof. Consider $f : X \longrightarrow Y$ is preopen, α-continuous and $W \in PO(X)$, then $f(W) \subset f(W^{-\circ})$ whenever $W^{-\circ}$ is an open set, so by preopenness of f we have $f(W^{-\circ}) \subset [f(W^{-\circ})]^{-\circ} \subset [f(W^{-})]^{-\circ}$ and by Lemma (4.1.) $f(W^{-\circ}) \subset [f(W)]^{=\circ} = [f(W)]^{-\circ}$ hence $f(W) \in PO(Y)$.

Corollary 4.1. If $f : X \longrightarrow Y$ is surjective preopen and α-continuous, then the image of each preclosed set in X is preclosed in Y.

Proof. Obvious by taking the complements in previous theorem.

Theorem 4.2. If $f : X \longrightarrow Y$ is surjective preopen and continuous, then the following verify :

(i) f is M-pre-open.

(ii) f is M-pre closed.

(iii) The image of each semi-open set is β-open.

(iv) The image of each semi-closed set is β-closed.

Proof. (i) Let $W \in PO(X)$, then $f(W) \subset f(W^{-\circ})$ by preopenness of f and since $W^{-\circ}$ is open, hence $f(W) \subset [f(W^{-\circ})]^{-\circ} \subset [f(W^{-})]^{-\circ} \subset [f(W)]^{=\circ} = [f(W)]^{-\circ}$.
Thus f is M-preopen.

(ii) Immediately.

(iii) Assume $V \in SO(X)$, then $f(V) \subset f(V^{\circ -})$ and so, $f(V) \subset [f(V^{\circ})]^{-}$ but V° is an open set hence $f(V) \subset [f(V^{\circ})]^{-\circ -}$

$\subset [f(V)]^{-\circ-}$ which gives that $f(V) \in \beta O(Y)$.
(iv) If F is semi-closed in X, then $X-F \in SO(X)$ by (iii) we get $f(X-F) \in \beta O(Y)$ therefore f(F) is β-closed in Y. Hence the result.

Corollary 4.2. Under the same hypothesis in the previous theorem if Y is extremely disconnected then the last two statements will be "The image of each semi-open (resp. semi-closed) is preopen (resp. preclosed).
Proof. Omitted by (i) and (ii) of Theorem (2.3).

Theorem 4.3. A β-continuous mapping which is a.o.s. is preopen.
Proof. Let $f : X \longrightarrow Y$ be β-continuous a.o.s. and U is an open in X. Since $U^{-\circ}$ is regular open in X, then $f(U^{-\circ})$ is open in Y. Thus $f(U) \subset f(U^{-\circ}) = [f(U^{\circ-\circ})]^{\circ} \subset [f(U)]^{-\circ}$ (by (v) of Theorem (2.1) in [1]). Hence the result.

Corollary 4.3. $f : X \longrightarrow Y$ is preclosed if f is β-continuous surjective and a.o.s.

Theorem 4.4. If $f : X \longrightarrow Y$ is surjective and Y is extremely disconnected, then f is preopen (resp. preclosed) whenever it is β-open (resp. β-closed).
Proof. Follows immediatelly by using (i) and (ii) of Theorem (2.3).

Delta J.Sci. 12 (2)1988

R. A. Mahmoud

Theorem 4.5. The following statements verify for the two mappings $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$.

(i) If f and g are preopen (resp. preclosed), f is surjective and $g(V^-) \subset [g(V)]^-$ for each set $V \subset Y$, then g is preopen (resp. preclosed).

(ii) If Y is extremely disconnected and both f and g are preopen (resp. preclosed surjective), then gof is also.

(iii) If gof and g are preopen (resp. preclosed surjective) and X is extremely disconnected, then f is A-continuous.

5. COMPARISONS

Next lemma is very important in this sequal :

Lemma 5.1. [10] If $f : X \longrightarrow Y$ is precontinuous and semi--continuous, then f is α-continuous.

Theorem 5.1. A β-continuous mapping from an extremely disconnected space into any space and semi-continuous it is α-continuous.

Proof. Follows directly by Theorem (3.3) and Lemma (5.1).

Theorem 5.2. For a mapping $f:X \longrightarrow Y$ if X is extremely disconnected, then the following statements are equivalent:

(i) f is continuous.

(ii) f is β-continuous and A-continuous.

(iii) f is β-continuous and LC-continuous.

Proof. (i) $\Longrightarrow$ (ii) and (i) $\Longrightarrow$ (iii) are obvious, but (ii) $\Longrightarrow$ (i) and (iii) $\Longrightarrow$ (i) by using Theorem (3.3) and Theorem (4) in [6].

The following theorem is an improvement of one result in [8].

Theorem 5.3. A precontinuous a.o.s. mapping $f : X \longrightarrow Y$ is preopen.

Proof. Let U be an open set in X. Since $U^{-\circ}$ is regular open, then by a.o.s. $f(U^{-\circ})$ is open and therefore we have $f(U) \subset f(U^{-\circ}) = [f(U^{-\circ})]^{\circ} \subset [f(U^{-})]^{\circ} = [f(U^{\circ -})]^{\circ}$. Since f is precontinuous, then $[f(U^{\circ -})]^{\circ} \subset [(f(U)]^{-\circ}$. Hence the result.

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خصص هذا البحث لدراسة خواص آخرى جديدة لهذا النوع الجديد من المجموعات ومنها كان ضروريا امتداد الدراسة الى الراسم المعتمد على هذه المجموعات واستنباط خواص جديده له وعلاقته بنظائره من الرواسم الاخرى قريبة الانفتاح ، هذا وقد قوينا عدد من النتائج التى كانت مستنبطه من قبل .