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ON THE SPECTRUM AND EXPANSION IN EIGENFUNCTIONS OF A SINGULAR DIFFERENTIAL OPERATOR

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ABSTRACT

In this paper we are dealing with the spectrum and expansion in eigenfunctions of a singular differential operator. According to a given class of certain solutions for an associated eigenvalue equation we investigate the discrete spectrum and the resolvent set of the operator. Furthermore, we study the continuous spectrum and obtain the resolvent. Lastly, we give, in terms of the eigenfunctions of the considered operator, the expansion of any function in the space $L_2(o, \infty)$ which satisfies particular conditions.

INTRODUCTION

The spectral theory of differential operators with discontinuous coefficient is a modern trend in advanced athematics. It should be mentioned that the problem of finding the frequencies of the free oscillations of a semi--infinite string of density $\rho = \rho(x)$ reduces to the differential equation $-y'' - \lambda \rho(x)y = o$, $x > o$ (see [3]). The spectrum and expansion in eigenfunctions of a singular differential operator of the second order is an intersted subject for many mathematician. Naimark [5] and Marchenko [4] studied a second order differential operator with boundary condition at the origin ($y(o) = o$). Petrosian [6] investigated the same problem when one of the coefficients of the operator is discontinuous. Darwish [1,2] was concerned with a singular differential operator having the non local boundary condition $y'(o) + \alpha \int_o^\infty K(x)\, y(x)\, dx = o$, our goal in this paper is to study a singular differential operator of the second order when the discontinuous coefficient takes complex values and $y'(o) = o$. We define a singular differential operator, denoted by L, and give certain solutions of an associated eigenvalue equation which play an important role through out this article. We investigate the discrete spectrum and the resolvent set of the operator. In addition, we study the continuous spectrum and construct the resolvent. Finally, we obtain the expansion of any function defined in the Hilbert space $L_2(o, \infty)$ which satisfies particular conditions in terms of the eigenfunctions of the considered operator.

1. The operator L.

In the space $L_2(o, \infty)$ let us consider the differential operator L generated by the differential expression

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$$\ell(y) = \frac{1}{\rho(x)} \left\{ - \frac{d^2 y}{d x^2} \right\}$$

and the condition $y'(o) = o$, where

$$\rho(x) = \begin{cases} a^2 , & o \leqslant x \leqslant 1, \quad \text{Im } a \neq o \\ 1 , & 1 < x < \infty . \end{cases}$$

Let D_L denotes the domain of definition of L consisting of all functions $y(x) \in L_2(o , \infty)$ which satisfy the following conditions :

i) The function $y'(x)$ exists and is absolutely continuous in every closed interval $[o, b]$, $b > o$,

ii) The differential expression $\frac{1}{\rho(x)} \left\{ -y'' \right\}$ is in $L_2(o , \infty)$,

iii) The function $y(x)$ satisfies the condition $y'(o) = o$.

We define the operator L by the formula $Ly = \ell(y)$ for all $y \in D_L$ and then the eigenvalue equation is $Ly = \lambda y$. Let $\lambda^{\frac{1}{2}}(=k)$ denotes $\sigma + i\tau$ such that $o < \arg k < \pi$.

2. Some certain solutions of the equation $(\ell - \lambda)y = o$

It is evident that the differential equation

$$y'' + \lambda \rho(x) y = o \tag{1}$$

is satisfied by $\varphi(x,k) := \cos kax$ and $\theta(x ,k) := \frac{\sin kax}{ka}$

which satisfy respectively the initial conditions

$$y(o) = 1 \ , \quad y'(o) = o \quad \text{and} \quad y(o) = o \quad , \quad y'(o) = 1 \qquad (2)$$

Since the initial conditions (2) do not depend on k, it follows that for every $x \geq o$, $\varphi(x, k)$ and $\theta(x, k)$ are entire functions of k [5] and they construct a fundamental system of solutions of equation (1) on (o, 1). In addition, we have on $(1, \infty)$ the following solutions of equation (1)

$$f(x,k) = e^{ikx} \quad \text{and} \quad f_1(x,k) = e^{-ikx} \ .$$

Further

$$W [\ f(x,k) \ , \ f_1(x,k)] = - 2ik \neq o. \qquad (I)$$

Thus, they construct a fundamental system of solutions on $(1, \infty)$.

Now, we continute the solution f(x,k) on (o, 1). For this purpose, suppose that the function f(x,k) is continuous and has a continuous first derivative at x=1.

Since the solutions $\varphi(x,k)$ and $\theta(x, k)$ construct a fundamental system of solutions on (o, 1) , then $f(x, k)$ can be represented on this interval as follows

$$f(x, k) = c_1(x)\varphi(x, k) + c_2(k)\,\theta(x, k).$$

Here, from the continuouty of $f(x, k)$ and $f'(x, k)$ at the point $x = 1$, we have

$$f(x, k) = e^{ik}\,[\cos ka(x-1) + \frac{i}{a}\sin ka(x-1)].$$

Consequently, $f(x, k)$ can be written on the half line (o, ∞) in the form

$$f(x, k) = \begin{cases} e^{ik}[\cos ka(x-1) + \frac{i}{a}\sin ka(x-1)]\,, & o \leqslant x \leqslant 1 \\ e^{ikx}\,, & 1 \leqslant x < \infty \end{cases} \qquad (3)$$

Similarly, we can continue the solution $f_1(x , k)$ on (o,1) to obtain

$$f_1(x, k) = \begin{cases} e^{-ik}[\cos ka(x-1) - \frac{i}{a} \sin ka(x-1)] & , \quad o \leqslant x \leqslant 1 \\ e^{-ikx} & , \quad 1 \leqslant x < \infty \end{cases} \qquad (4)$$

on the half line (o, ∞). Let us now extend the solution $\varphi(x, k)$ on the interval $(1, \infty)$. Since $f(x, k)$, $f_1(x, k)$ construct fundamental solutions on the interval $(1, \infty)$, thus we can write $\varphi(x, k) = d_1(k) f(x, k) + d_2(k) f_1(x,k)$, where $d_1(k)$ and $d_2(k)$ are determined from the condition that $\varphi(x, k), \varphi'(x, k)$ are continuous at $x = 1$. Thus, we have

$$\varphi(x, k) = \cos ka \cos k(x-1) - a \sin ka \sin k(x-1)$$

and accordingly the function $\varphi(x, k)$ can be written on the half line as follows

$$\varphi(x, k) = \begin{cases} \cos kax & , \quad o \leqslant x \leqslant \\ \cos ka \cos k(x-1) - a \sin ka \sin k(x-1), & 1 \leqslant x < \end{cases} \qquad (5)$$

3. The discrete spectrum of the operator

Theorem 1 : The operator L has no positive eigenvalues.

For the proof see Naimark ([5.p. 126]).

Here, assume that λ is not on the positive semi axis, thus f(x, k) (formula (3)) is in $L_2(o, \infty)$ and $f_1(x, k)$ (formula (4)) is not. Hence, we have the following theorem :

Theorem 2 : The eigenvalues of the operator L are given by solutions of $W(k) := f'(o, k) = o$ in the upper half plane. The following theorem deals with the zeros of $W(k) = f'(o, k)$.

Theorem 3 : The set of eigenvalues of the operator L is no more than countable and its limit points can lie only on the half axis $\lambda \geq o$. These eigenvalues are all simple .

Proof : From formula (3), it is evident that the function $W(k) := f'(o, k)$ is holomorphic in the upper half plane and $f'(o, k) \neq o$ for all real $k \neq o$, then the set of its zeros is no more than countable and having o as the only possible limit point. Now, we prove that the zeros of the function $W(k) = f'(o, k)$ are all simple. To do this, we find an expression for $\frac{d}{dk} W(k) = \dot{W}(k)$. On differentiating the equation

$$f''(x, k) + k^2 \rho(x) f(x, k) = o \qquad (6)$$

with respect to k , we have

$$\dot{f}''(x, k) + k^2 \rho(x) \dot{f}(x, k) + 2k \rho(x) f(x, k) = o \quad (7)$$

From (6) , (7) we obtain

$$[\dot{f}''(x, k) f(x, k) - f''(x, k) \dot{f}(x, k)] + 2k \rho(x) [f(x, k)]^2 = o$$

which can be reformulated as :

$$\frac{d}{dx} w \left\{ f(x, k), \dot{f}(x, k) \right\} + 2k \rho(x) [f(x, k)]^2 = o.$$

Integrating this equality with respect to x from b to c , we get

$$w \left\{ f(x, k), \dot{f}(x, k) \right\} \Big|_b^c + 2k \int_b^c \rho(x) [f(x, k)]^2 \, dx = o. \quad (8)$$

Let $k = i\gamma$, ($\gamma > o$) be a zero of the function

W(k) (= f'(o, iγ)= o) and this implies that f(x, iγ) is real valued. On the other hand, one has

$$\lim_{x \to o} w \left\{ f(x, i\gamma), f(x, i\gamma) \right\} = f(x, i\gamma)\, \dot{f}'(x, i\gamma)$$

and

$$\lim_{x \to \infty} w \; f(x, i\gamma), \dot{f}(x, i\gamma) = o.$$

Consequently, it yields from (8) that

$$f(x, i\gamma)\, \dot{f}'(x, i\gamma) = 2i\gamma \int_o^{\infty} \rho(x)\, [f(x, i\gamma)]^2\, dx \neq o$$

and whence $\dot{f}'(x, i\gamma) \neq o$,

Thus, the zeros of the function w(k) are all simple and this completes the proof of the theorem.

Now, we study the assymptotic behaviour of the eigenvalues of the differential operator L. By virtue of the formula (3) we have

$$f'(o, k) = e^{ik}(ka \sin ka + ik \cos ka) , \quad \tau \geq o.$$

Hence, and according to Theorem 2 the eigenvalues of the operator L are the roots, located in the upper half plane, of the equation

$$a \sin ka + i \cos ka = o$$

which can be written in the form

$$\frac{a+1}{a-1} = e^{2ika} \qquad (9)$$

Theorem 4 : If $a = s + it$, $s \neq o$, $t \neq o$, then the operator L has a countable set of complex eigenvalues $\{\lambda_n\}$, where

$$\lambda_n = k_n^2 = \left\{ \frac{1}{2(si - t)} \left[\ln \left| \frac{s + it + 1}{s + it - 1} \right| + 2\pi ni + i \arg \left(\frac{s + it + 1}{s + it - 1} \right) \right] \right\}^2$$

Proof : From (9) we have

$$k = \frac{1}{2ia} \ln \frac{a+1}{a-1} \text{ and whence } k_n = \frac{1}{2ia} \left[\ln \left| \frac{a+1}{a-1} \right| + 2\pi ni + i \arg \left(\frac{a+1}{a-1} \right) \right].$$

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Since, $a = s + it$, $s \neq o$, $t \neq o$, then

$$k_n = \frac{1}{2i\ (s + it)}\ [\ \ln \left|\frac{s + it + 1}{s + it - 1}\right| + 2\pi ni +$$

$$i \arg \left(\frac{s + it + 1}{s + it - 1}\right)]$$

consequently

$$\lambda_n = k_n^2 = \left\{\frac{1}{2(si - t)}\ [\ln \left|\frac{s + it + 1}{s + it - 1}\right| + 2\pi ni +\right.$$

$$\left. i \arg \left(\frac{s + it + 1}{s + it - 1}\right)]\right\}$$

which completes the proof.

4. The resolvent and continuous spectrum of the operator L.

Theorem 5 : (i) If λ ($= k^2$) is not an eigenvalue of the operator L, then the Green's function $G(x , \xi , k)$ for $\ell(y) - \lambda y = f$ is $R(x , \xi ,k)$ on the interval $o \leqslant x < \infty$; namely if $f \in L_2(o , \infty)$ then $y = R_\lambda (f) = \int_o^\infty R(x, \xi ,k) \times \rho(\xi)\ f(\xi)\ d\xi$, where

$$R(x,\xi,k) = \begin{cases} \dfrac{f(x,k)\, f_1'(o,k)\, f(\xi,k)}{2ik\ w(k)} - \dfrac{1}{2ik} f(x,k) f_1(\xi,k), & \xi \leqslant x \\ \dfrac{f(x,k)\, f_1'(o,k)\, f(\xi,k)}{2ik\ w(k)} - \dfrac{1}{2ik} f_1(x,k) f(\xi,k), & \xi \geqslant x. \end{cases} \qquad (10)$$

(ii) All λ $(=k^2)$ belongs the resolvent set of the operator L whenever $W(k) \neq o$ and $\tau > o$

Proof : The resolvent R_λ exists according to the assumption that λ $(=k^2)$ is not an eigenvalue of the operator L. If $R_\lambda(f) = y$, we have $Ly = f$. This means that there exists a solution of the equation $\ell(y) - \lambda y = f$ belongs to $L_2(o, \infty)$ and satisfies the condition $y'(o) = o$. By virtue of (I) it follows that the functions $f(x, k)$ and $f_1(x, k)$ are linearly independent solutions of the homogeneous equation $\ell(y) - \lambda y = o$. Making use of the method of variation of parameters we obtain the following general solution of $\ell(y) - \lambda y = f$

$$y(x,k) = A(x)\ f(x,k) + B(k)\ f_1(x,k)-$$

$$- \frac{1}{2ik} f(x,k) \int_0^x f_1(\xi,k)\, \rho(\xi)\, f(\xi)\, d\xi - \frac{1}{2ik} f_1(x,k) \int_x^{\infty} f(\xi,k)\, \rho(\xi)\, f(\xi)\, d\xi. = A(k)\, f(x,k) + B(k)\, f_1(x,k) + \int_0^{\infty} K(x,\xi,k)\, \rho(\xi)\, f(\xi)\, d\xi,$$

where $A(k) = c_1(o\ ,k)$, $B(k) = c_2(\infty,k)$ and

$$K(x,\xi,k) = \frac{-1}{2ik} \begin{cases} f(x\ ,\ k)\ \ f_1(\xi\ ,\ k) & ,\ \ \xi < x \\ f_1(x\ ,\ k)\ f(\xi\ ,\ k) & ,\ \ \xi > x. \end{cases}$$

It is easy to see that $|K(x , \xi , k)| < \exp[-\tau\, |x-\xi|]$ for $\tau > o$ and whence the integral $\int_0^{\infty} K(x\ ,\ \xi\ ,\ k)\, \rho(\xi)\, f(\xi)\, d\xi$ is bounded by convolution of a function in $L_1(o\ ,\ \infty)$ and a function in $L_2(o\ ,\ \infty)$ and thus this integral can be represented by a function belongs to $L_2(o\ ,\ \infty)$. On the other hand, since $f(x,k) \in L_2(o\ ,\ \infty)$ and $f_1(x,k) \overline{\in}\, L_2(o\ ,\ \infty)$, $\tau > o$ then we can take $B(k) = o$. Consequently we get

$$y(x\ ,\ k) = A(k)\ f(x\ ,\ k) + \int_{o}^{\infty} K(x\ ,\ \xi\ ,\ k)\ \rho(\xi)\ f(\xi) d\xi .$$

Now, since $y'(o\ ,\ k) = o$, we have

$$A(k)\ f'(o\ ,\ k) - \frac{1}{2ik}\ f_1'\ (o\ ,\ k) \int_{o}^{\infty} f(\xi ,k)\ \rho(\xi)\ f(\xi)\ d\xi = o$$

which yields

$$A(k) = \frac{f_1'(o\ ,\ k)}{2ik\ f'(o\ ,\ k)} \int_{o}^{\infty} f(\xi ,\ k)\ \rho(\xi)\ f(\xi)\ d\xi .$$

Hence,

$$y(x,k) = \frac{f(x,k) f_1'(o,k)}{2ik\ f'(o,k)} \int_{o}^{\infty} f(\xi ,\ k)\ \rho(\xi)\ f(\xi)\ d\xi + \int_{o}^{\infty} K(x\ ,\ \xi\ ,k) \rho(\xi)\ f(\xi)\ d\xi = \int_{o}^{\infty} R(x, \xi\ ,k) \rho(\xi)\ f(\xi)\ d\xi ,$$

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where

$$R(x,\xi,k) = \frac{f_1(x,k)\,f_1'(o,k)\,f(\xi,k)}{2ik\,w(k)} + K(x,\xi,k).$$

Consequentely, we obtain

$$R(x,\xi,k) = \begin{cases} \dfrac{f(x,k)\,f_1'(o,k)f(\xi,k)}{2ik\,w(k)} - \dfrac{1}{2ik}\,f(x,k)\,f_1(\xi,k), & \xi \leqslant x \\[2ex] \dfrac{f(x,k)\,f_1'(o,k)f(\xi,k)}{2ik\,w(k)} - \dfrac{1}{2ik}\,f_1(x,k)\,f(\xi,k), & \xi \geqslant x. \end{cases}$$

Thus, (i) is completely proved; the proof of (ii) follows by using theorem 2.

Theorem 6: Every point of the semi-axis $\lambda \geqslant o$ is a point of the spectrum of the operator L.

Proof : Let $Q(\xi,k) = \frac{-1}{2ik} f_1'(o,k)\,f(\xi,k)$ and substitute in (10) to have

$$R(x,\xi,k) = \frac{-Q(\xi,k)\,f(x,k)}{w(k)} - \frac{1}{2ik}\,f_1(\xi,k)\,f(x,k)$$

as $\xi \leqslant x$.

Let us now define the following function

$$F(x\ ,\ k) = \begin{cases} \bar{f}_1(x,k) - \dfrac{\bar{Q}(x,k) \int_0^a \bar{f}_1(\xi\ ,k)\ Q(\xi\ ,k)d\xi}{\int_0^a |Q(\xi\ ,k)|^2\ d\xi}, & x \leqslant a \\ 0 & x > a \end{cases}$$

From above we obtain

$$\int_0^a F(x,k)\ Q(x,k)dx = \int_0^a \bar{f}_1(x,k)\ Q(x,k)dx - \int_0^a \bar{Q}(x,k)Q(x,k)dx$$

$$\cdot \int_0^a \bar{f}_1(\xi\ ,k)Q(\xi\ ,k)d\xi \Big/ \int_0^a |Q(\xi\ ,k)|^2\ d\xi$$

$$= \int_0^a \bar{f}_1(x,k)Q(x,k)dx - \int_0^a |Q(x,k)|^2\ dx$$

$$\cdot \int_0^a \bar{f}_1(\xi\ ,k)Q(\xi\ ,k)d\xi \Big/ \int_0^a |Q(\xi\ ,k)|^2\ d\xi$$

$$= \int_0^a \bar{f}_1(x,k)Q(x,k)dx - \int_0^a f_1(\xi\ ,k)Q(\xi\ ,k)d\xi$$

$$= 0$$

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and using this result in the following integral yields

$$\int_0^a F(x,k)\,\bar{F}(x,k)dx = \int_0^a F(x,k)\, f_1(x,k)dx.$$

Consequently we determine $\| R_\lambda \;F(x\;,\;k)\|^2$ as follows :

$$R_\lambda \;F(x,k) = \int_0^\infty \frac{-Q(\xi\;,k)\;f(x,k)\;F(\xi\;,k)}{w(k)}\;d\xi$$

$$- \frac{1}{2ik} \int_0^\infty f_1(\xi\;,k)\;f(x,k)\;F(\xi\;,k)\;d\xi$$

$$= \int_0^a - \frac{f(x,k)}{2ik}\, f_1(\xi\,,k)\;F(\xi\;,k)d\xi$$

$$= - \frac{f(x,k)}{2ik} \int_0^a f_1(\xi\;,k)\;F(\xi\;,k)\;d\xi$$

Hence,

$$\| R_\lambda \;\; F(x,k)\;\|^2 = \int_0^\infty | R_\lambda \;\; F(x,k)\;|^2\;dx \geq \int_a^\infty |R_\lambda\, F(x,k)|^2\;dx$$

$$= \int_a^{\infty} \left| -\frac{f(x,k)}{2ik} \int_o^a \left|F(\xi\ ,k)\right|^2 d\xi \right|^2 dx$$

$$= \frac{1}{4|k|^2} \int_a^{\infty} |f(x,k)|^2\ dx \int_o^a |F(\xi\ ,k)|^4\ d\xi .$$

Since $f(x,k) = \exp(ikx)$ as $x \in [1, \infty)$ there exists a sufficient large a such that $|f(x,k)| > \frac{1}{2} \exp(-\tau x)$ for $a < x < \infty$, $\tau > o$ and $k \neq o$. Thus, $\int_o^{\infty} |f(x,k)|^2\ dx \geq \frac{1}{8\tau} \exp(-2\tau a)$. Since in any semi circle in the upper half plane with centre at the origen the integral $\int_o^a |F(x,k)|\ dx$ is bounded away from zero, then $\|R_\lambda\|^2 \geq \frac{c\ e^{-2\tau a}}{4|k|^2\ 8\tau}$, where c is a constant. From this, it follows that $\|R_\lambda\| \to \infty$ as $\tau \to o$ and so the square of the point is in the spectrum of the operator L (i.e., $\lambda(= k^2) \geq o$) and this completes the proof of the theorem.

<u>Theorem 7 :</u> The continuous spectrum of the operator L lies on the half semi-axis $\lambda \geq o$.

Proof : We have to show that the domain of the resolvent operator, i.e. the range of the operator $(L - \lambda I)$, is dense in $L_2(o, \infty)$. This is equivalent to prove that the orthogonal complement of this range which coincides with the space of solutions of $L^* z = \lambda \bar{\rho} z$ is the zero element [5]. It is evident that the adjoint operator L^* of L can be defined by the following problem $-z'' = \lambda \bar{\rho} z$, $z'(o) = o$. Conversely, suppose that there is a function $z \in L_2(o, \infty)$ which is different from zero such that $(Ly, z) = (y, L^* z)$. Hence, for $\lambda \geq o$ there exists a non-trivial solution of the equation $(y, L^* z) = o$ which belongs to $L_2(o, \infty)$. Since by theorem 1 the operator does not have positive eigenvalues, then there is no non trivial solution of $(y, L^* z) = o$ belonging to $L_2(o, \infty)$ and whence the assertion follows.

5. The expansion in eigenfunction of the operator L :

Theorem 8 : If the function $\psi(x) \in L_2(o, \infty)$ is finite in both neighbourhoods of $x=o$, $x=\infty$ and has a continuous second derivative in $L_2(o, \infty)$, then

$$\int_o^\infty R(x, \xi, k) \rho(\xi) \psi(\xi)\, d\xi = -\frac{\psi(x)}{k^2} + \frac{1}{k^2} \int_o^\infty R(x, \xi, k) g(\xi) d\xi \qquad (11)$$

where $g(\xi) = -\psi''(\xi)$.

Moreover, if $\tau > 0$, $k \longrightarrow \infty$ we have

$$\int_0^{\infty} R(x,\xi,k)\,\rho(\xi)\,\psi(\xi)\,d\xi = -\frac{\psi(x)}{k^2} + O\left(\frac{1}{k^2}\right). \quad (11')$$

Proof : By virtue of the formula (10), we have

$$\int_0^{\infty} R(x,\xi,k)\,\rho(\xi)\,\psi(\xi)\,d\xi = \left[\int_0^{x} R(x,\xi,k) + \int_x^{\infty} R(x,\xi,k)\right]\rho(\xi)\,\psi(\xi)\,d\xi$$

$$= \int_0^{x}\left[\frac{f(x,k)\,f_1'(o,k)\,f(\xi,k)}{2ik\;\;w(k)} - \frac{f(x,k)f_1(\xi,k)}{2ik}\right]\rho(\xi)\psi(\xi)\,d\xi$$

$$+ \int_x^{\infty}\left[\frac{f(x,k)f_1'(o,k)\,f(\xi,k)}{2ik\;w(k)} - \frac{f_1(x,k)\,f(\xi,k)}{2ik}\right]\rho(\xi)\psi(\xi)\,d\xi$$

Thus,

$$\int_{o}^{\infty} R(x,\xi,k)\,\rho(\xi)\,\psi(\xi)\,d\xi = \int_{o}^{\infty}\left[\frac{f(x,k)\ f_1'(o,k) f(\xi,k)}{2ik\ w(k)}\right] \rho(\xi)\psi(\xi)d\xi - \int_{o}^{x} \frac{f(x,k)\ f_1(\xi,k)}{2ik}\rho(\xi)\psi(\xi)d\xi - \int_{x}^{\infty}\frac{f_1(x,k) f(\xi,k)}{2ik}\rho(\xi)\psi(\xi)d\xi$$

Since $f(x,k)$ and $f_1(x,k)$ are solutions of equation (1), we have

$$\int_{o}^{\infty} R(x,\xi,k)\,\rho(\xi)\,\psi(\xi)\,d\xi = \frac{f(x,k)f_1'(o,k)}{2ik\ w(k)} \int_{o}^{\infty}\left[\frac{-1}{k^2}\ f''(\xi,k)\right]\psi(\xi)d\xi - \frac{f(x,k)}{2ik}\int_{o}^{\infty}\left[-\frac{f_1''(\xi,k)}{k^2}\right]\psi(\xi)d\xi - \frac{f_1(x,k)}{2ik}\int_{x}^{\infty}\left[\frac{-f(\xi,k)}{k^2}\right]\psi(\xi)d\xi \,. \qquad (12)$$

Intergrate twice the right-hand side of (12) to get

$$\int_0^\infty R(x,\xi,k)\,\rho(\xi)\psi(\xi)\,d\xi = \frac{f(x,k)f_1'(o,k)}{2ik\,w(k)}\int_0^\infty f(\xi,k)\,\psi''(\xi)\,d\xi$$

$$+\frac{f(x,k)\psi(x)f_1'(x,k)}{2ik^3} - \frac{f(x,k)\psi'(x)f_1(x,k)}{2ik^3}$$

$$+\frac{f(x,k)}{2ik^3}\int_0^x f_1(\xi,k)\psi''(\xi)\,d\xi$$

$$-\frac{f_1(x,k)\psi(x)f'(x,k)}{2ik^3}$$

$$+\frac{f_1(x,k)f(x,k)\psi'(x)}{2ik^3}$$

$$+\frac{f_1(x,k)}{2ik^3}\int_x^\infty f(\xi,k)\psi''(\xi)\,d\xi$$

$$= -\frac{\psi(x)[f_1(x,k)f'(x,k) - f(x,k)f_1'(x,k)]}{2ik^3}$$

$$+ \frac{1}{k^2} \int_o^{\infty} \frac{f(x,k)f_1'(o,k)f(\xi ,k)}{2ik\, w(k)} \psi''(\xi)d\xi$$

$$- \int_o^{x} \frac{f(x,k)f_1(\xi ,k)\ \psi''(\xi)d\xi}{2ik}$$

$$- \int_x^{\infty} \frac{f_1(x\ ,\ k)\ f(\xi\ .\ k)}{2ik}\ \psi''(\xi)d\xi$$

Making use of the above obtained results to have

$$\int_o^{\infty} R(x, \xi ,k)\, \rho(\xi)\, \psi(\xi)d\xi = - \frac{\psi(x)}{k^2} + $$

$$+ \frac{1}{k^2} \int_o^{\infty} R(x, \xi ,k)g(\xi)d\xi ,$$

where

$$g(\xi) = - \psi''(\xi).$$

Moreover, if $\tau > o$, $|k| \to \infty$, we get $R(x, \xi ,k)= O(1)$

and therefore $\int_{o}^{\infty} R(x,\xi,k)\, \rho(\xi)\, \psi(\xi)\, d\xi =$

$$= -\frac{\psi(x)}{k^2} + O\left(\frac{1}{k^2}\right).$$

<u>Theorem (9) :</u> If the function $\psi(x)$ satisfies the conditions of theorem 8, then the expansion of a function $\psi(x) \in L_2(o,\infty)$ in eigenfunctions of the operator L takes the form

$$\psi(x) = \frac{1}{\pi} \int_{o}^{\infty} k\, dk \int_{o}^{\infty} \frac{u(x,k)\, v(\xi,k)}{w(k)\, w(-k)}\, \rho(\xi)\psi(\xi)d\xi$$

$$+ \sum_{n=1}^{\infty} 2\, k_n \int_{o}^{\infty} \frac{f(x, k_n)\, v(\xi,k_n)}{w(k_n)}\, \rho(\xi)\psi(\xi)d\xi. \qquad (13)$$

where, $u(x,k) = f(x,-k)\, w(k) - f(x,k)\, w(-k)$,

$$v(\xi,k) = \frac{f_1(\xi,k)\, f'(o,k) - f(\xi,k) f_1'(o,k)}{2ik}$$

Proof :

From the assumption the equality (11') holds and by multiplying it by $\frac{k}{\pi i}$ we have

$$\frac{2k}{2\pi i}\int_0^\infty R(x,\xi,k)\,\rho(\xi)\,\psi(\xi)\,d\xi = -\frac{\psi(x)}{\pi ik} + O\left(\frac{1}{k}\right).$$

We integrate this equality along the closed contour $\Gamma_{r,\varepsilon}$ $(=S_{r,\varepsilon}\cup \ell_{r,\varepsilon})^{*}$ and we obtain

$$\frac{1}{2\pi i}\oint_{\Gamma_{r,\varepsilon}} 2k\,dk\int_0^\infty R(x,\xi,k)\,\rho(\xi)\,\psi(\xi)\,d\xi = -\psi(x) + \oint_{\Gamma_{r,\varepsilon}} O\left(\frac{1}{k}\right)dk \quad (14)$$

Since the function $R(x,\xi,k)$ is analytic in the upper half plane, therefore we get

$$\frac{1}{2\pi i}\oint_{\Gamma_{r,\varepsilon}} 2k\,dk\int_0^\infty R(x,\xi,k)\,\rho(\xi)\,\psi(\xi)\,d\xi = I^1_{r,\varepsilon} + I^2_{r,\varepsilon}\,,\quad \text{where}$$

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*) $S_{r,\varepsilon}$ is a part from the semi circle S_r of radius r and center at zero, and $\ell_{r,\varepsilon}$ is the line $\{k;\ \tau=\varepsilon\}$.

$$I^1_{r,\varepsilon} = \frac{1}{2\pi i} \int_{S_{r,\varepsilon}} 2k\ dk \int_0^\infty R(x, \xi, k)\, \rho(\xi)\, \psi(\xi) d\xi, \tag{15}$$

$$I^2_{r,\varepsilon} = \frac{1}{2\pi i} \int_{l_{n\varepsilon}} 2k\ dk \int_0^\infty R(x, \xi, k)\, \rho(\xi)\, \psi(\xi) d\xi. \tag{16}$$

By letting $r \longrightarrow \infty$ in (14) and using both (15) and (16) we find

$$\psi(x) = \frac{1}{\pi i} \int_0^\infty k dk \int_0^\infty \{R(x, \xi, k + i\varepsilon) - - R(x, \xi, k - i\varepsilon)\}\, \rho(\xi)\, \psi(\xi) d\xi \tag{17}$$

From (10) it follows for $k \in (-\infty, -r) \cup (r, \infty)$ that

$$\lim_{\varepsilon \to o} \int_0^\infty R(x, \xi, k \pm i\varepsilon)\, \rho(\xi)\, \psi(\xi) d\xi = \int_0^\infty R(x, \xi, k \pm io)\, \rho(\xi)\, \psi(\xi) d\xi$$

exists and that for $k \in (-r, r)$ the integral

$\int_{o}^{\infty} R(x,\xi,k)\,\rho(\xi)\,\psi(\xi)\,d\xi$ respresents a meromorphic function with simple poles located at the zeros $k = k_n$, $|k| < r$ of $f'(o\ , k)$. Hence, from (17) we have

$\psi(x) = \psi_1 + \psi_2$, where

$$\psi_1(x) = \frac{1}{\pi i} \int_{o}^{\infty} k\ dk \int_{o}^{\infty} \{R(x,\xi,k+io)-R(x,\xi,k-io)\}\ \rho(\xi)\,\psi(\xi)\,d\xi$$

and

$$\psi_2 = - \sum_{n=1}^{\infty} \text{Res}\ (2k \int_{o}^{\infty} R(x,\xi,k)\,\rho(\xi)\,\psi(\xi)\,d\xi \Big|_{k=k_n} .$$

To determine ψ_1 let

$$V(\xi,k) = \frac{f_1(\xi,k)\ f'(o,k) - f(\xi,k)\ f_1'\ (o,k)}{2ik} ,$$

thus $V(o,k) = 1$, $V'(o,k) = o$. Thus, by virtue of the formula (10) we have

$$R(x, \xi ,k) = \frac{-1}{w(k)} f(x,k)\, V(\xi ,k) \quad \text{for } \xi \leq x.$$

Now, since $V(o,k) = 1$ and $V'(o,k) = o$, then

$$R(x, \xi ,k+io) - R(x, \xi ,k-io) = \left\{ \frac{1}{w(-k)} f(x,-k) - \frac{1}{w(k)} f(x,k) \right\} V(\xi ,k)$$

$$= \left\{ \frac{f(x,-k)w(k) - f(x,k)w(-k)}{w(k)\, w(-k)} \right\} V(\xi ,k)$$

$$= \frac{U(x , k)}{w(k)\, w(-k)}\, V(\xi ,k),$$

where

$$U(x , k) = f(x , -k)\, w(k) - f(x , k)\, w(-k).$$

Thus, we have

$$\psi_1 = \frac{1}{\pi} \int_0^{\infty} \left\{ \int_0^{\infty} [R(x, \xi ,k+io) - R(x, \xi ,k-io)] \right.$$

$$\rho(\xi)\psi(\xi)d\xi\Big\} k\,dk = \frac{1}{\pi}\int_0^\infty k\,dk \int_0^\infty \frac{U(x,k)\;V(\xi,k)}{w(k)\;w(-k)}\,\rho(\xi)\psi(\xi)d\xi \qquad (18)$$

For $\xi \geq x$ the calculations yield the same result as (18).

Carring out the contour integration in ψ_2 we get

$$-\sum_{n=1}^{\infty} \mathrm{Res}\left(2k\int_0^\infty R(x,\xi,k)\,\rho(\xi)\psi(\xi)d\xi\right|_{k=k_n}$$

$$= \sum_{n=1}^{\infty} 2k_n \int_0^\infty \frac{f(x,k_n)\;V(\xi,k_n)}{\dot{w}(k_n)}\,\rho(\xi)\psi(\xi)d\xi \qquad (19)$$

Hence, from (18) and (19) we obtain the following expansion

$$\psi(x) = \frac{1}{\pi}\int_0^\infty k\,dk\int_0^\infty \frac{U(x,k)\;V(\xi,k)}{w(k)\;w(-k)}\,\rho(\xi)\psi(\xi)d\xi$$

$$+\sum_{n=1}^{\infty} 2k_n \int_0^\infty \frac{f(x,k_n)\;V(\xi,k_n)}{w(k_n)}\,\rho(\xi)\psi(\xi)d\xi\,.$$

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" استنتاج الطيف والمفكوك بدلالة الدوال الذاتية لمؤثر غاضلى شاذ "

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