Delta J. Sci. (11)(3) 1987 994-1006

SOME PROPERTIES OF m* - FUZZY MEASURABLE SETS AND CONSTRUCTION OF FUZZY OUTER MEASURES

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Received : 30 - 8 - 1987

ABSTRACT

After Zadeh [1] introduced the notion of fuzzy sets. E. P. Klement [2],[3] introduced the concept of fuzzy measures as a generalization of classical fuzzy measure [4]. It is the purpose of this paper to continue the work started in [5] and give some properties of m* - fuzzy measurable sets. Also, we give nessecary and sufficient conditions for a fuzzy set to be fuzzy measurable with respect to some known fuzzy outer measure m* . We introduce the definition of a fuzzy sequential covering class of a set X . We prove that we can construct a fuzzy outer measure from any fuzzy sequential covering class K of X and any extended real valued non-negative function f with domain K, such that f(0) = o .

1- INTRODUCTION

Throughout this paper, X will denote an ordinary set and the unit interval [0,1] will be denoted by I. The complement of a set $A \subset X$ will be denoted by A^c and the complement of a fuzzy set u will be denoted by u^c, i.e . $u^c = 1 - u$. The fuzzy set 1_A , $A \subset X$ is defined by

$$1_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \in A^c \end{cases} .$$

The definitions of fuzzy σ - algebras and measures were introduced by E. P. Klement in [3] as follows.

1- A subfaily $\sigma \subset I^X$ is called <u>fuzzy σ -algebra</u> iff it satisfies the following conditons:

(σ1) $\forall$ constant a : $a \in \sigma$;
(σ2) $\forall\, u \in \sigma$: $u^c \in \sigma$;
(σ3) $\forall\, (u_n)_{n \in N} \in \sigma^N$: $\bigvee_{n \in N} u_n \in \sigma$,

where N is a countable set.

(Clearly , if σ is a fuzzy σ -algebra, (σ3) holds when N is a finite set).

2- A functional $m : \sigma \longrightarrow R_+^*$, where R_+^* is the set of all extended non-negative real numbers, is called a <u>fuzzy measure</u> on (X, σ) iff it fulfills the following properties:

(M 1) the domain σ of m is a fuzzy σ -algebra ;

(M 2) $m(0) = 0$;

(M 3) $\forall\, u, v \in \sigma : m(u \vee v) + m(u \wedge v) = m(u) + m(v)$;

(M 4) $\forall\, (u_n)_{n \in N} \in \sigma^N : \; u_n \uparrow u \Longrightarrow m(u_n) \uparrow m(u)$,

The definitions fuzzy outer measures and m^* -fuzzy measurable sets were introduced in our work [1] .

Definition 1. A functional $m^* : I^X \longrightarrow R_+^*$ is called a fuzzy outer measure on X iff fulfills the following properties :

(01) the domain of m^* is the set of all fuzzy sets;

(02) $m^*(0) = 0$;

(03) $\forall\, u, v \in I^X , \; u \leqslant v \Longrightarrow m^*(u) \leqslant m^*(v)$;

(04) $\forall\, (u_n)_{n \in N} \in I^X \Longrightarrow m^*(\bigvee_{n \in N} u_n) \leqslant \sum_{n \in N} m^*(u_n)$,

where N is a countable set.

(Notice that, if m^* is a fuzzy outer measure on X , then (04) holds when N is a finite set).

Definition 2. Let m^* be a fuzzy outer measure on X. Then a fuzzy set u is said to be m^* -fuzzy measurable iff for any arbitary fuzzy set μ , we have

$$m^*(\mu) = m^*(\mu \wedge 1_{\{x \in X : u(x) \leqslant c\}}) + m^*(\mu \wedge 1_{\{x \in X : u(x) > c\}}) =$$

$$= m^*(\mu \wedge 1_{\{x \in X: u(x) < c\}}) + m^*(\mu \wedge 1_{\{x \in X: u(x) \geq c\}}), \tag{1}$$

for every constant $c \in I$.

II. Properties of m^* -fuzzy measurable sets

In the following theorem, we study some properties of m^* -fuzzy measurable sets, where m^* is some fuzzy outer measure on X.

Theorem 1. Let m^* be a fuzzy outer measure on X and let u be an m^* -fuzzy measurable set. Then for any arbitrary fuzzy set μ, we have

$$\begin{aligned} m^*(\mu \wedge 1_{\{x \quad X: u(x) \leq c\}}) &= m^*(\mu \wedge 1_{\{x \in X: u(x) < c\}}) + \\ &+ m^*(\mu \wedge 1_{\{x \in X: u(x) = c\}}), \end{aligned} \tag{2}$$

and

$$\begin{aligned} m^*(\mu \wedge 1)_{\{x \in X: u(x) \geq c\}}) &= m^*(\mu \wedge 1)_{\{x \in X: u(x) > c\}}) + \\ &+ m^*(\mu \wedge 1_{\{x \in X: u(x) = c\}}). \end{aligned}$$

Proof. Let u be an m^* -fuzzy measurable set . Then u

satisfies (1) for every constant $c \in I$ and any arbitrary fuzzy set u.
Thus , we have

$$m^*(\mu \wedge 1_{\{x \in X:u(x) \leq c\}}) =$$

$$= m^*((\mu \wedge 1_{\{x \in X:u(x) \leq c\}}) \wedge 1_{\{x \in X:u(x) < c\}}) +$$

$$+ m^*((\mu \wedge 1_{\{x \in X:u(x) \leq c\}}) \wedge 1_{\{x \in X:u(x) \geq c\}}) =$$

$$= m^*(\mu \wedge 1_{\{x \in X:u(x) < c\}}) + m^*(\mu \wedge 1_{\{x \in X:u(x) = c\}}),$$

which proves (2).
In a similar way , (3) can be proved.

<u>Theorem 2.</u> Let m^* be a fuzzy outer measure on X. A fuzzy set u is m^*-fuzzy measutable iff for any arbitrary fuzzy set μ and for every constant $c \in I$, we have

$$m^*(\mu) = m^*(\mu \wedge 1_{\{x \in X:u(x) < c\}}) + m^*(u \wedge 1_{\{x \in X:u(x) > c\}}) + m^*(u \wedge 1_{\{x \in X:u(x) = c\}}). \qquad (4)$$

<u>Proof.</u> First, let u be an m^*-fuzzy measurable set.

Then for every constant $c \in I$ and any arbitrary set μ, it follows from definition 1, that

$$m^*(\mu) = m^*(\mu \wedge 1_{\{x \in X:\, u(x) \leq c\}}) + m^*(\mu \wedge 1_{\{x \in X: u(x) > c\}}) =$$

$$= m^*(\mu \wedge 1_{\{x \in X: u(x) < c\}}) + m^*(\mu \wedge 1_{\{x \in X: u(x) \geq c\}}) .$$

Applying theorem 1, we obtain

$$m^*(\mu) = m^*(\mu \wedge 1_{\{x \in X: u(x) < c\}}) + m^*(\mu \wedge 1_{\{x \in X: u(x) > c\}}) +$$

$$+ m(\mu \wedge 1_{\{x \in X: u(x) = c\}}) .$$

Now, for every constant $c \in I$ and each arbitrary fuzzy set μ, let u be a fu⁊zy set satisfying (4).
From the subadditivity of m^*, we have

$$m^*(\mu \wedge 1_{\{x \in X: u(x) \leq c\}}) \leq m^*(\mu \wedge 1_{\{x \in X: u(x) < c\}}) +$$

$$+ m^*(\mu \wedge 1_{\{x \in X: u(x) = c\}}) .$$

Hence from (4),

$$m^*(\mu) \geq m^*(\mu \wedge 1)_{\{x \in X: u(x) \leq c\}}) + m^*(\mu \wedge 1_{\{x \in X: u(x) > c\}}). \tag{5}$$

But,

$$\mu = (\mu \wedge 1_{\{x \in X:(x) \leq c\}}) \vee (\mu \wedge 1_{\{x \in X:u(x) > c\}}),$$

Then from the subadditivity of m^*,

$$m^*(\mu) \leq m^*(\mu \wedge 1_{\{x \in X:u(x) \leq c\}}) + m^*(\mu \wedge 1_{\{x \in X:u(x) > c\}}). \tag{6}$$

From (5) and (6), we obtain

$$m^*(\mu) = m^*(\mu \wedge 1_{\{x \in X:u(x) \leq c\}}) + m^*(\mu \wedge 1_{\{x \in X:u(x) > c\}}). \tag{7}$$

In a similar way, we can prove that

$$m^*(\mu) = m^*(\mu \wedge 1_{\{x \in X:u(x) < c\}}) + m^*(\mu \wedge 1_{\{x \in X:u(x) \geq c\}}). \tag{8}$$

From (7) and (8), it follows that u is m^*-fuzzy measurable and the proof is complete.

<u>Theorem 3.</u> Let m^* be a fuzzy outer measure on X. The fuzzy set 1_A, where A is some subset of X, is m^*-fuzzy measurable iff for each arbitrary fuzzy set μ, we have

$$m^*(\mu) = m^*(\mu \wedge 1_A) + m^*(\mu \wedge 1_{(A^c)}) . \qquad (9)$$

<u>Proof.</u> Let 1_A , $A \subset X$ is m^* -fuzzy measurable. Then

$$m^*(\mu) = m^*(\mu \wedge 1_{\{x \in X: 1_A(x) \leq c\}}) + m^*(u \wedge 1_{\{x \in X: 1_A(x) > c\}}),$$

for every constant $c \in I$.
By taking $c = 0$, we obtain

$$\{x \in X: 1_A(x) \leq 0\} = 1_{(A^c)} ,$$

$$\{x \in X: 1_A(x) > 0\} = 1_A ,$$

then

$$m^*(\mu) = m^*(\mu \wedge 1_A) + m^*(\mu \wedge 1_{(A^c)}) .$$

Now, let (9) is satisfied, then for every constant $c \in I$ and any arbitrary fuzzy set μ, we have

(i) in the case $c = 1$, we get

$$m^*(u \wedge 1_{\{x \in X: 1_A(x) \leq c\}}) + m^*(\mu \wedge 1_{\{x \in X: 1_A(x) > c\}}) =$$

$$= m^*(\mu \wedge 1_X) + m^*(\mu \wedge 1_\phi) = m^*(\mu) .$$

(ii) in the case $c < 1$,

$$m^*(\mu \wedge 1_{\{x \in X: 1_A(x) \leq c\}}) + m^*(u \wedge 1_{\{x \in X: 1_A(x) > c\}}) =$$

$$= m^*(\mu \wedge 1_{(A^c)}) + m^*(\mu \wedge 1_A) = m^*(\mu).$$

Hence for every $c \in I$, we have

$$m^*(\mu) = m^*(\mu \wedge 1_{\{x \in X: 1_A(x) \leq c\}}) + m^*(\mu \wedge 1_{\{x \in X: 1_A(x) > c\}}).$$

Similarly, by studying the cases $c = o$ and $c > 0$, we can prove that

$$m^*(\mu) = m^*(\mu \wedge 1_{\{x \in X: 1_A(x) < c\}}) + m^*(\mu \wedge 1_{\{x \in X: 1_A(x) \geq c\}}).$$

Thus , 1_A is m^* -fuzzy measurable and the proof is complete.

III. Construction of fuzzy outer measures

First , we introduce the following definition.

Definition 3. Let K be a class of fuzzy sets. We call K a sequential fuzzy covering class of X if

(i) $o \in K$;

(ii) for every fuzzy set u, there is a sequence $\{w_n\}$ in K such that

$$u < \bigvee_{n=1}^{\infty} w_n .$$

Now, let f be an extended real valued non-negative function with domain K such that $f(0) = 0$.

For each fuzzy set u, let

$$m^*(u) = \inf \left[\sum_{n=1}^{\infty} f(w_n) : w_n \in K , \bigvee_{n=1}^{\infty} w_n \geq u \right]. \qquad (1)$$

Theorem 2. For any sequential fuzzy covering class K and for any non-negative extended real valued function f with domin K and with $f(0) = 0$, the function m^* defined by (1) is a fuzzy outer measure .

Proof. (01) , (02) and (03) are obviously satisfied.

It remains to verify (04) .

Let $\{u_n\}$ be any sequence of fuzzy sets and let u be any positive number . For each positive integer n there is a sequence $\{w_{nj}\}$ in K such that

$$u_n \leq \bigvee_{j=1}^{\infty} w_{nj} ,$$

$$\sum_{j=1}^{\infty} f(w_{nj}) \leq m^*(u_n) + \frac{\varepsilon}{2^n} . \qquad (2)$$

Then , we have

$$\bigvee_{n=1}^{\infty} u_n \leq \bigvee_{n,j=1}^{\infty} w_{nj} .$$

Hance by (1) and (2), we get

$$m^*\left(\bigvee_{n=1}^{\infty} u_n\right) \leqslant \sum_{n,j=1}^{\infty} f(w_{nj}) \leqslant$$

$$\leqslant \sum_{n=1}^{\infty} \left[m^*(u_n) + \frac{\varepsilon}{2^n} \right] \leqslant$$

$$\leqslant \sum_{n=1}^{\infty} m^*(u_n) + \varepsilon .$$

Since ε is arbitrary , we get

$$m^*\left(\bigvee_{n=1}^{\infty} u_n\right) \leqslant \sum_{n=1}^{\infty} m^*(u_n) , \qquad (3)$$

and the proof is complete.

Corollary. Theorems (1) and (2) enable us to construct a fuzzy measure from any given pair K,f.

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بعض الخواص للمجموعات القياسية الفازية * m
وبناء قياسات فازية خارجية

د. رفعت حسنين عطية

بعد تعريف زاده لمفهوم المجموعة الفازية أدخل كليمين مفهوم الفازية كتعميم للمفهوم التقليدى للقياسات الفازية كتعميم للمفهوم التقليدى للقياسات . الغرض من هذا البحث هو الاستمرار فى نفس اتجاة البحث رقم ٥ واعطاء بعض الخواص للمجموعات الفازيه الـ m^* قياسية . ايضا أعطينا الشروط الكافية والضرورية لكى تصبح المجموعه الفازيه ذات قياس فازى بالنسبة لبعض الـ m^* قياسات الفازيه الخارجية المعروفة ايضا أدخلنا تعريف الغطاء الفازى التتابعى للمجموعة X . واثبتنا أنه يمكننا تكوين قياس فازى خارجى من أى غطاء فازى تتابعى لـ K من X وأى داله ممتدة غير سالبة وحقيقية القيمه ولها النطاق K بحيث $f(0) = 0$