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# ON HALPERN'S CONJECTURE IN SPACE FORMS

**BY**

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**ABSTRACT**

In [ 1 ] , Halpern conjectured some relations between bitangent lines, double points and inflection points of a smooth closed plane curve . Ozawa [ 2 ] proved that Halpern's conjecture in still true for curves in hyperbolic as well as spherical space.

## 1- INTRODUCTION

Firstly, we give introductory material concerning smooth curves in a plane.

Let $c : S^1 \longrightarrow R^2$ be a smooth closed plane curve given in general position. A bitangent of c is a line which is tangent to c at only two differnt points. A bitangent T is said to be exterior or interior, according as in small neighborhoods of the two points of tangency of T to c , the curve c lies on the same side or on opposite sides of T. Let II (c) and I (c) denote the numbers of exterior and interior bitangents of c, respectively.

let $B(c) = II(c) + I(c)$ and $D(c)$ denote the numbers of all bitangents and all double points of c, respectively. It is possible that the curvature K of the curve c at a point t is zero ($K(t) = o$). If, in addition, $K^{\cdot}(t) \neq o$ (and hence the zero of K is isolated ) c(t) is called an inflection point of the curve [3]. In other words, a point $t \in c$ is an inflection point if the curve c changes from convex to concave around t ( or the opposite). Under this condition, the curvature K should vanish at this point.

In this paper we require that all bitangents, double points and inflection points are of the regular type. By the term " regular" we mean that:

(a) For a plane curve $c(t)$, we say that the points $c(u)$ and points $c(u)$ and $c(v)$ are regular tangent pair if $c(u) \neq c(v)$, $c'(u) = c'(v)$ and $K(u) \neq o$ , $K(v) \neq o$.

(b) A double point is regular if the curve crosses itself transnormally (in an acute angle).

(c) An inflection point is regular if it an isolated zero of the curvature K.

These conditions (a),(b) and (c) we call the regularity conditions. In this paper all curves we are dealing with satisfy these three conditions unless we mention other properties.

In (1962), Fabricuis-Bjerre [4] discovered the equation

$$II(c) = I(c) + D(c) + F(c)/2, \quad (1)$$

which represents a necessary but not sufficient condition [5] for a plane curve.

In (1970), Halpern [1] gave this question; what is the necessary and sufficient condition for integers II, I, D and F/2 to be attained by the integers II(c), I(c), D(c) and F(c)/2 for some curve c in general position(*)? and conjectured that :

(i) any quadruplet (II, I, D, F/2) satisfying (1) with $F/2 \geq 1$ can be attained by a curve.

(ii) a quadruplet (II, I, D, F/2) with F=o should satisfy the inequality $I \leq D^2 - D$ and the equality $I \equiv o \pmod 2$ besides (1) in order to be attained by a curve.

Remarks.

1) When $F > o$, the curve should not be convex.

2) For a convex closed curve, it is necessary that the number of regular inflection points F=o, but this is not a sufficient condition. See figure (1).

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*) For the general position concept, see [2].

(a) non-convex (F=o) (b) convex (F=o)

Fig. (1).

In (1984), Ozawa [2] proved that Halpern's conjecture is true, and gave the answer for the above mentioned question in the following theorem :

<u>Theorem (1-1)</u>

Under the regularity conditions (a),(b) and (c), the necessary and sufficient condition for non-negative integers II, I , D and F/2 to be attained by a smooth curve c as

$$(II, I, D, F/2) = (II(c), I(c), D(c), F(c)/2)$$

is the following

(a) $II = I + D + F/2$ , and

(b) if $F = o$ , $I \leqslant D^2 - D$ and I is even.

We give the following example to show that the regularity conditions are necessary for theorem (1-1) to be satisfied.

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Example

In figure (2), we find that, I=o , II=1 , D=o , F=o. Therefore the equation

$$II = I + D + F/2$$

is not applicable. The reason is that there exist bitangents, say T, touching the curve at more than two different points. Also, the curve contains a lot of irregular inflection points.

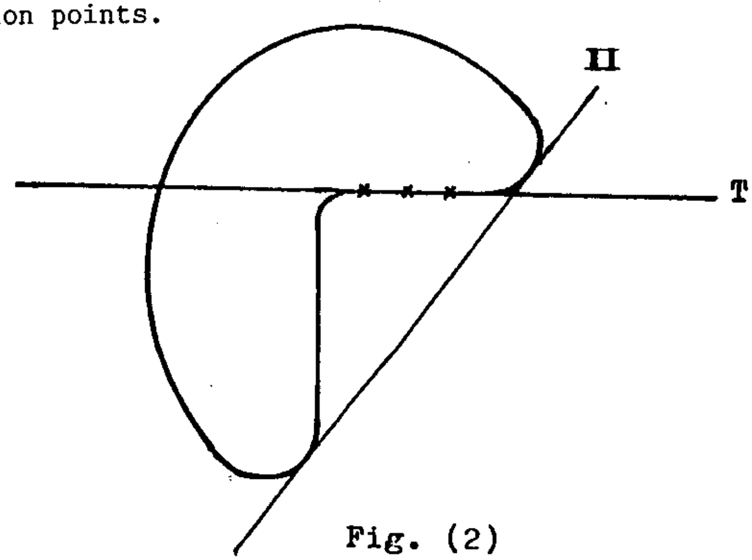


Fig. (2)

The main aim of this paper is to show that Halpern's conjecture and theorem (1-1) have analogue forms in 2-dimensional hyperbloic space $H^2$ and the open hemispere $S^2_{+}$ when replacing exterior and interior bitangent lines by exterior and interior bitangent geodesics as being defined below.

Consequently, we prove the following

Theorem (1-2).

Under the regularity conditions (a) , (b) and (c), the necessary and sufficient condition for non-negative integers II, I, D and F/2 to be attained by a smooth cuve $\alpha : S^1 \longrightarrow H^2$ as

$$(II, I, D, F/2) = (II(\alpha), I(\alpha), D(\alpha), F(\alpha)/2)$$

is the following :

(a) $II = I + D + F/2$, and

(b) if $F = o$ , $I \leqslant D^2 - D$ and I is even.

Theorem (1-3).

Under the regularity conditions (a), (b) and (c), the necessary and sufficient condition for non-negative integers II, I, D and F/2 to be attained by a smooth curve $\gamma : S^1 \longrightarrow S^2_\nu$ as

$$(II, I, D, F/2) = (II(\gamma) , I(\gamma), D(\gamma), F(\gamma)/2)$$

is the following :

a) $II = I + D + F/2$ and

b) if $F = o$ , $I \leqslant D^2 - D$ and I is even.

to realize this work we need the following section.

## 2. Beltrami map.

A complete, simply connected , $C^\infty$, Riemannian manifold of constant negative sectional curvatures is called a hyperbolic space. Let us consider the spherical model of the n-dimensional hyperbolic space $H^n = \{(x^o, \ldots, x^n) \in V^{n+1} : -(x^o)^2 + \ldots + (x^n)^2 = -1,\ x^o > o\ \}$, where $V^{n+1}$ is the Minkowski space [1] . We can show that the Minkowski metric is Riemannian when restricted to $H^n$, moreover the sectional curvature of $H^n$ is $-1$.

Now, let us mention some geometric properties of an important map which is useful in solving the considered problem. This map is called the central projection map (or Beltrami map) . The central projection map $\beta : H^n \longrightarrow E^n$ is defined to be the map which takes x from n-dimensional hyperbolic space $H^n$ to the intersection of n-dimensional Euclidean space $E^n$ (tangent to $H^n$ at p ) with the straight line through x and the origin o of the Minkowski space $V^{n+1}$ [1] .

The map $\beta$ takes $H^n$ diffeomorphically to the open disk $D(p,1) \subset E^n$ of centre P and radius 1. See figure(3). From the geometric point of view $\beta$ is a geodesic map. Consequently, $\beta$ takes a totally geodesic m-submanifold of $H^n$ to a linear subvariely $E^m$ of $E^n$. Also if a submanifold $M \subset H^n$ is locally convex then so is $\beta(M) \subset E^n$ .

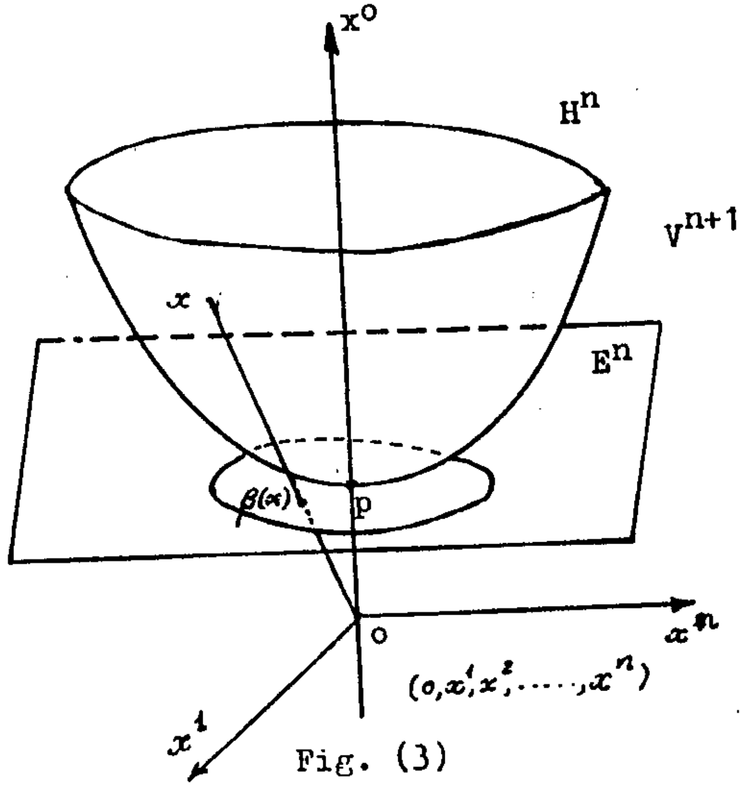


Fig. (3)

Also, the map takes hypersurfaces of $H^n$ with sectional curvature $^*K \geqslant -1$ to hypersurfaces of $E^n$ with sectional curvature $K \geqslant o$, [3].

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*) The sectional curvature is defined only for the manifolds with dimensions $n \geqslant 2$, but if $n = 1$, is taken to be the totally curvature of the curve.

From the geometric properties of the map $\beta : H^2 \longrightarrow E^2$. we prove the following :

Lemma.

The maps $\beta$ and $\beta^{-1}$ preserve the inflection points.

Proof.

Let $\alpha$ be a curve in $H^2$, $s \in \alpha$ be an inflection point i.e., $K(s) = -1$ and the curve $\alpha$ changes, at this point, from convex to concave (or the opposite). By using the properties of the map $\beta$, we see that $K(\beta(s)) = o$ and the curve $\beta(\alpha)$ in $E^2$ change, at the point $\beta(s)$, from convex to concave ( or the opposite). Thus the point $\beta(s)$ is an inflection point for the curve $\beta(\alpha)$. Since the map $\beta^{-1}$ has the same properties as $\beta$, then $\beta^{-1}$ perserves the inflection points. See figure (4).

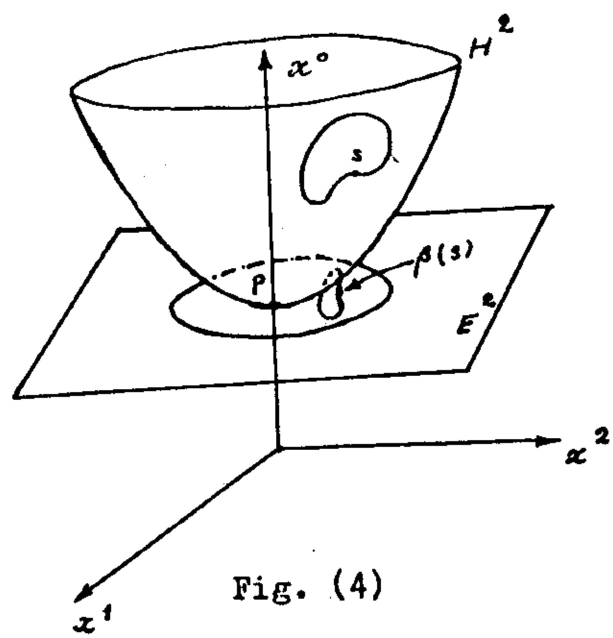


Fig. (4)

## 3- MAIN WORK

At first, we try to extend all the above geometric properties of plane curves to include curves in hyperbolic space.

For a smooth closed curve $\alpha : S^1 \longrightarrow H^2$ given in general position in 2-dimensional hyperbolic space $H^2$, a bitangent geodesic of $\alpha$ is a geodesic in $H^2$ which is tangent to $\alpha$ at only two different points. A bitangent geodesic T is said to be exterior or interior, according as in small neighborhoods of the two points of tangency of T with $\alpha$, the curve $\alpha$ lies on the same side or an opposite sides of T. Let $II(\alpha)$ and $I(\alpha)$ denote the numbers of exterior and interior bitangent geodesies of $\alpha$, respectively. Let $B(\alpha)$ and $D(\alpha)$ denote the numbers of all bitangent geodesies and all double points of $\alpha$, respectively ($B(\alpha) = II(\alpha) + I(\alpha)$). Since $\alpha$ is in general position, then the number $B(\alpha)$ of bitangent geodesies is always finite. Asmooth curve $\alpha$ has generically a finite number $D(\alpha)$ of double points and no triple points, nor more, and a finite even number $F(\alpha)$ of inflection points. These facts can be easily proved through applying the projection maps $\beta$ and $\beta^{-1}$.

Now, we shall use the central projection map to transfer the problem under consideration from $H^2$ to $E^2$.

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Therefore, by using $\beta : H^2 \longrightarrow D(p,1) \subset E^2$ , we have the composition $\beta \circ \alpha : S^1 \longrightarrow E^2$ which represents a smooth closed plane curve in $E^2$ . See figure (5).

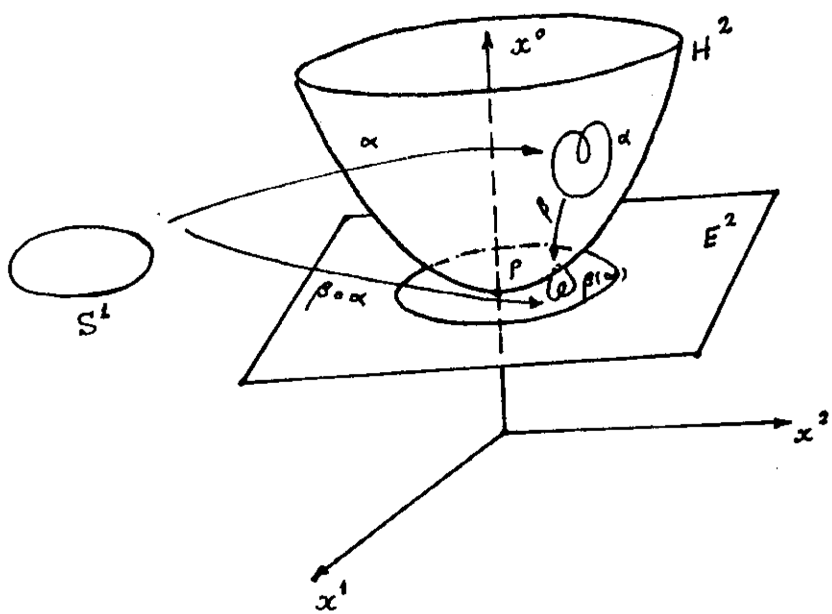


Fig. (5)

Then Halpern's conjecture and the theorem (1-1) are true for the curve $\beta \circ \alpha$ [2]. As we know that the inverse map $\beta^{-1}$ has the same properties as $\beta$ [6], then the integers I , II, D ,F/2 are the same for both $\alpha$ and $\beta(\alpha)$ . Thus, inequalities and relations between these numbers for $\alpha$ are exactly the same as those of $\beta(\alpha)$. Consequently, Halpern's conjecture and the theorem (1-2) are true in $H^2$.

In [1], Halpern expected that his conjecture is true in the 2-dimensional sphere $S^2$ but he did not give any suppot to his expectation. If we confine ourselves to the study of curves in an open hemisphere, we can show easily that Halpern's conjecture is true. Our tool in proving this result is the using of the central projection map- as it has been just used for the hyperbolic space-to transfer the whole problem under consideration from $S^2$ to $E^2$ (See Fig. (6)).

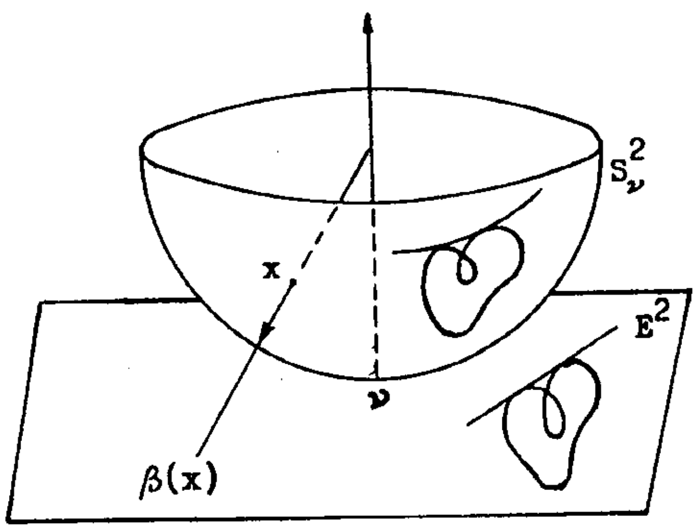


Fig. (6).

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# دراسة لتوقع هالبيرن فى الفراغات ثابته التقوس

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توقع الرياضى هالبيرن فى عام ١٩٧٠ من خلال دراسة العديد من الأمثلة الهندسيه تحقق نتيجة رياضيه تربط بين عدد المماسات والنقط الثنائيه ونقط الانقلاب لأى منحنى أملس مغلق فى مستوى ديكارتى .

وفى عام ١٩٨٤ تمكن الرياضى أوزاوا من تقديم برهان رياضى جيد تبين من خلاله صحة ما توقعه هالبيرن سابقا .

وفى بحثنا هذا نقدم محاوله ناجحه لاثبات أن توقع هالبيرن يظل صحيحا بالنسبه الى المنحنيات الملساء المغلقه فى الفراغات ذات التقوس الثابت موجبا كان هذا التقوس أو سالبا .