Delta J. Sci(11)(3)1987 1007 - 1016

FUZZY MEASURES AND FUZZY OUTER MEASURES CONSTRUCTED BY MEANS OF SOME KNOWN FUZZY OUTER MEASURES

BY

REFAAT HASSANEIN ATTIA

Math. Dep. Faculty of Science. Sohag Branch of
Assiut University. Sohag. Egypt

Received: 30 - 8 -1987

ABSTRACT

In this paper, we construct new fuzzy outer measures from some given fuzzy outer measures. In the last theorem of this paper , we give a sufficient condition on a fuzzy outer measure to be a fuzzy measure.

1- INTRODUCTION

Throughout this paper, the following notations will be used. X will denote an ordinary set which we fix throughout the paper. I is the closed unit interval $[0,1]$. R^*_+ [1]. is the set of all extended non-negative real numbers. The complement of a fuzzy set u will be denoted by u^c , i.e. $u^c = 1 - u$. The crisp fuzzy set 1_A , $A \subset X$ is defined by

$$1_A(x) = \begin{cases} 1 & \text{if } x \in A \\ o & \text{if } x \in A^c \end{cases} .$$

The definition of fuzzy σ - algebra and fuzzy measures were introduced by E. P. Klement in [2] , as follows.

1- A <u>fuzzy σ -algebra</u> is defined as a subfamily $\sigma \subset I^X$, satisfying the following conditions:

(σ1) $\forall$ constant a : $a \in \sigma$;

(σ2) $\forall u \in \sigma$; $u^c \in \sigma$;

(σ3) $\forall (u_n)_{n \in N} \in \sigma^N$, N is a countable set $\Longrightarrow \sup_{n \in N} u_n \in \sigma$.

(Notice that if σ is a fuzzy σ -algebra , then (σ3) holds when N is a finite set).

2- A <u>fuzzy measure</u> is defined as a functional $m : \sigma \longrightarrow R_+^*$ fulfilling the following properties:

(MI) the domain σ of m is a fuzzy σ -algebra ;

(M2) $m(0) = 0$;

(M3) $\forall u,v \in \sigma$: $m(u \vee v) + m(u \wedge v) = m(u) + m(v)$;

(M4) $\forall (u_n)_{n \in N} \in \sigma^N$: $u_n \uparrow u \Longrightarrow m(u_n) \uparrow m(u)$.

The definition of fuzzy outer measure was first introduced in our work [3], this definition is needed in this paper. (For the orinary measure see [4]

Delta J. Sci. (11)(3)1987

Definition 1. A functional $m^{*} : I^{X} \longrightarrow R_{+}^{*}$ is called a fuzzy outer measure on X iff the following conditions are satisfied:

(01) the domain of m^{*} is all the fuzzy sets on X ;
(02) $m^{*}(0) = 0$;
(03) $\forall u, v \in I^{X}$, $u \leq v : m^{*}(u) \leq m^{*}(v)$;
(04) $\forall (u_n)_{n \in N} \in \sigma^{N}$, N is a countable set:

$$m^{*}\left(\bigvee_{n \in N} u_n\right) \leq \sum_{n \in N} m^{*}(u_n).$$

(Clearly , a fuzzy outer measure is finitely subadditive i.e. (04) holds when N is a finite set).

II. Fuzzy outer measures constructed by means of some given fuzzy outer measures

In the following two theorems,we show how to construct fuzzy outer measures from given fuzzy outer measures.

Theorem 1. If m^{*} a fuzzy outer measure on X and v is a fixed fuzzy set, then the function μ^{*} with domain I^{X} given by

$$\mu^{*}(u) = m^{*}(u \wedge v)$$

is a fuzzy outer measure on X .

Proof. Clearly u^* satisfies (01) and (o2).
Let u,w be any two fuzzy sets such that $u \leq w$, we have

$$\mu^*(u) = m^*(u \wedge v) \leq m^*(w \wedge v) = \mu^*(w).$$

Thus μ^* satisfies (03).
To show that μ^* satisfies (04), let $\{u_n : n = 1,2,3,\dots\}$ be a sequence of fuzzy sets. We have

$$\mu^*\left(\bigvee_{n=1} u_n\right) = m^*\left(\left(\bigvee_{n=1} u_n\right) \wedge v\right) m^*\left(\bigvee_{n=1} (u_n \wedge v)\right) \leq$$

$$\leq \sum_{n=1} m^*(u_n \wedge v) = \sum_{n=1} \mu^*(u_n),$$

which complete the proof of the theorem.

Theorem 2. If $\{m^*_n\}$ is a sequence of fuzzy outer measures on X, then $\sum_{n=1} m^*_n$ defined by

$$\left(\sum_{n=1} m^*_n\right)(u) = \sum_{n=1} m^*_n(u),$$

is also a fuzzy outer measure on X.

Proof.. Let $\mu^* = \sum_{n=1} m^*_n$.
Clearly μ^* satisfies (01) and (02).

Delta J.Sci. (11)(3)1987

Assume u,w be any fuzzy sets such that $u \leqq w$.
Then

$$\mu^*(u) = (\sum_{n=1} m_n^*)(u) = \sum_{n=1} m_n^*(u) \leqq \sum_{n=1} m_n^*(w) =$$

$$= (\sum_{n=1} m_n^*)(w) = \mu^*(w) \quad ,$$

that is μ^* satisfies (03).
Let $\{ u_n \ : 1,2,3,\ldots \}$ be any sequence of fuzzy sets, we have

$$\mu^*(\bigvee_{k=1} u_k) = (\sum_{n=1} m_n^*)(\bigvee_{k=1} u_k) = \sum_{n=1} m_n^*(\bigvee_{k=1} u_k) \leqq$$

$$\leqq \sum_{n=1} \sum_{k=1} m_n^*(u_k) = \sum_{k=1} \sum_{n=1} m_n^*(u_k) =$$

$$= \sum_{k=1} (\sum_{n=1} m_n^*)(u_k) = \sum_{k=1} \mu^*(u_k) \ .$$

Therefore μ^* satisfies (04) and the proof is complete.

III. Sufficient condition for a fuzzy outer measure to be a fuzzy measure

In the following theorem, we give a sufficient condition for a fuzzy outer measure to be a fuzzy measure.

<u>Theorem 3.</u> Let m^* be a fuzzy outer measure on X. If m^* is additive i.e. for each $u,v \in I^X$ with $u \wedge v = 0$;

$$m^*(u \quad v) = m^*(u) + m^*(v),$$

then m^* is a fuzzy measure.

<u>proof.</u> It is clear that m^* satisfies (MI) and (M2). Now, let $u,v \in I^X$, then

$$m^*(u \vee v) = m^*((u \vee v) \wedge 1_{\{x \in X \,:\, u(x) \leq v(x)}) +$$

$$+ m^*(u \vee v) \wedge 1_{\{x \in X \,:\, u(x) > v(x)\}}) =$$

$$= m^*(v \wedge 1_{\{x \in X \,:\, u(x) \leq v(x)\}}) +$$

$$+ m^*(u \wedge 1_{\{x \in X \,:\, u(x) > v(x)}). \qquad (1)$$

$$m^*(u \wedge v) = m^*((u \wedge v) \wedge 1_{\{x \in X : u(x) \leq v(x)\}}) +$$

$$+ m^*((u \wedge v) \quad 1_{\{x \in X : u(x) > v(x)\}}) =$$

$$= m^{*}(u \wedge 1_{\{x \in X : u(x) \leqq v(x)\}}) +$$

$$+ m^{*}(v \wedge 1_{\{x \in X : u(x) > v(x)\}}). \qquad (2)$$

Addition (1) and (2) , we obtain

$$m^{*}(u \vee v) + m^{*}(u \wedge v) = [\, m^{*}(u \quad 1_{\{x \in X : u(x) > v(x)\}}) +$$

$$+ m^{*}(u \wedge 1_{\{x \in X : u(x) \leqq v(x)\}}) \,] +$$

$$+ [m^{*}(v \wedge 1_{\{x \in X : u(x) \leqq v(x)\}}) +$$

$$+ m^{*}(v \wedge 1_{\{x \in X : u(x) > v(x)\}})] =$$

$$= m^{*}(u) + m^{*}(v).$$

Thus , m^{*} satisfies (M3).
It remains to prove that m^{*} satisfies (M4).
Let $\{ u_n ; n = 1,2,3,\ldots\}$ be a sequence of fuzzy sets such that $u_n \leqq u_{n+1}$ for all $n = 1,2,3,\ldots$.
Define the fuzzy sets $\{ v_n ; n = 1,2,3,\ldots \}$ as follows:

(i) for $n = 1$: $v_1 = u_1 \wedge 1_{\{x \in X : u_1(x) < u_2(x)\}}$,

(ii) for $n > 1$: $v_n = u_n \wedge 1_{\{x \in X : u_{n-1}(x) = u_n(x) < u_{n+1}(x)\}}$.

obviously, $v_i \wedge v_j = 0$ for $i \neq j$; $i,j = 1,2,3,....$.
Moreover,

$$\bigvee_{i=1}^{n} v_i = u_n \wedge 1_{\{x \in X : u_n(x) < u_{n+1}(x)\}} ,$$

$$\bigvee_{i=1}^{\infty} v_i = \bigvee_{i=1}^{\infty} u_i = u .$$

Clearly $u_n \geqq \bigvee_{i=1}^{n} v_i$, then

$$m^*(u_n) \geqq m^*\Big(\bigvee_{i=1}^{n} v_i\Big) = \sum_{i=1}^{n} m^*(v_i) .$$

Thus , as $n \longrightarrow \infty$,

$$\lim_{n \to \infty} m^*(u_n) \geqq \sum_{i=1}^{\infty} m^*(v_i) \geqq m^*\Big(\bigvee_{i=1}^{\infty} v_i\Big) =$$

$$= m^*\Big(\bigvee_{i=1}^{\infty} u_i\Big) \quad m^*(u) . \qquad (3)$$

On the other hand,

$$u_n \leq \bigvee_{i=1}^{\infty} u_i = u ,$$

then $m^*(u_n) \leq m^*(u)$.

Hence

$$\lim_{n \to \infty} m^*(u_n) = m^*(u) . \qquad (4)$$

From (3) and (4) , we have

$$\lim_{n \to \infty} m^*(u_n) = m^*(u) .$$

Thus m^* satisfies (04) and the proof is complete.

REFERENCE

1- L.A. Zadeh: Fuzzy sets. Iform. and Control. 8 (1965), 338-353.

2- E.P. Klement: Fuzzy σ-algebras and fuzzy measurable functions. Fuzzy Sets and Systems. 4 (1980), 83 - 93 .

3- R. H. Attia: Fuzzy outer measures and fuzzy measurable sets. To appear.

4- P. R. Halmos: Measure Theory. (Van Nostrand, New York, 1968)

القياسات الغازية والقياسات الغازية الخارجية المكونه بواسطة قياسات فازية خارجية معلومة

د. رفعت حسانين عطية

فى هذا البحث نقوم بتكوين قياسات فازية خارجية جديدة باستخدام قياسات فازية خارجية معلومة . فى النظرية الاخيرة من ذلك البحث نقدم شروطا كافيه على القياس الفازى الخارجى ليكون قياسا فازيا .