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THE EFFECT OF ANHARMONIC TERMS ON THE RESONANCE OF AN OSCILLATING SYSTEM

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ABSTRACT

This paper deals with the investigation of the effect of anharmonic terms in a forced vibrating system when the absorbed energy from the external periodic force is maximum.

INTRODUCTION

It is known that an oscillating system under forced vibration is governed by the equation

$$\ddot{x} + \omega_0^2 x + \alpha' \dot{x} = p \cos \omega t \ .$$

It is usual to use non-dimensional units , $\omega_0 t$ for the time $\frac{p}{\omega_0^2}$ for the length , the above equation becomes

$$\ddot{x} + x + \alpha \dot{x} = \cos \omega t \qquad (\alpha = \frac{\alpha'}{\omega_0}) \ . \qquad (1)$$

The solution of this equation is well known

$$x = C \cos (\omega t - \epsilon)$$

$$\epsilon = \frac{\pi}{2} \qquad\qquad C = \frac{1}{\sqrt{(1-\omega^2)^2+\alpha^2\omega^2}} \tag{2}$$

The power absorbed is given by

$$P = \langle \dot{x} \cos \omega t \rangle = 1/2\ C\, \omega \sin \epsilon \tag{3}$$

and using the values of C, ω given by (1)

$$P = \frac{1}{2\sqrt{\alpha^2 + \left(\frac{1-\omega^2}{\omega}\right)^2}} \tag{4}$$

and this is maximum for

$$\omega^2 = 1 \tag{5}$$

giving for the maximum average power absorbed for the ideal oscillator

$$P = \frac{1}{2\alpha} \qquad \text{and} \qquad C = \frac{1}{\alpha} \tag{6}$$

It is now required to find the effect of adding an anharmonic term to the oscillating system ,

$$\ddot{x} + \omega_0^2 x + \alpha' \dot{x} + \lambda' x^3 = p \cos \omega t$$

Using the same untils as before, $\omega_0 t$ for the time $\frac{p}{\omega_0^2}$

for the length , we obtain the equation in dimensionless form

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$$\ddot{x} + x + \alpha \dot{x} + \lambda x^3 = \cos \omega t \qquad (7)$$

where $\lambda = \lambda^2 \frac{p^2}{\omega_0^6}$, $\alpha = \frac{\alpha'}{\omega_0}$.

It is found that for values of λ ranging between 0 and $2x\ 10^{-8}$ the behaviour is the same as in the case $\lambda = 0$, i.e the oscillating system has a powerful power absorption from the external field in resonance.

For higher values of λ although numerically very small has a big effect on the oscillating system and its power absorption at resonance as will be shown in the following.

Method of solution:

We now use the known approximate method to solve this equation. Assume the solution to given in terms of four paramaters

$$x = C \cos (\omega t - \epsilon) + C' \cos (3 \omega t - \epsilon') \qquad (8)$$

Substituting this solution in Eq. (7), we get

$$E(t) = \ddot{x} + x + \alpha \dot{x} + \lambda x^3 - \cos \omega t$$

$$= [-1+(1-\omega^2)C \cos\epsilon + \alpha\omega C \sin \epsilon + \frac{3}{4}\lambda (C^3 + 2CC'^2) \cos\epsilon +$$

$$+ \frac{3\lambda}{4} C^2 C' \cos (-2\epsilon + \epsilon')] \cos \omega t + [(1-\omega^2)C \sin\epsilon -$$

$$- \alpha\omega C \cos\epsilon + \frac{3\lambda}{4} (C^3 + 2CC'^2) \sin \epsilon +$$

$$+\frac{3}{4}C^2C'\sin(\;'-2\;)\,]\sin\;t+[(1-9\;^2)C'\cos\;'+$$

$$+3\;C'\sin\;'+\frac{}{4}C^3\cos 3\;+\frac{3}{4}(C'^3+2C^2C')\cos\;'\,]\cos 3$$

$$+[\,(1-9\;^2)\,C'\sin\;'-3\;C'\cos\;'+\frac{}{4}C^3\sin 3\;+$$

$$+\frac{3}{4}(C'^3+2C^2C')\sin\;'\,]\sin 3\;t+$$

$$+\frac{3}{4}[C^2C'\cos(2\;+\;')+C\,C'^2\cos(2\;'-\;)]\cos 5\;t+$$

$$+\frac{3\lambda}{4}[\,C^2C'\sin(2\epsilon+\epsilon')+C\,C'^2\sin(2\epsilon'-\epsilon)]\sin 5\omega t+$$

$$+\frac{3\lambda}{4}C\,C'^2\cos(2\epsilon'+\epsilon)\cos 7\omega t+\frac{3\lambda}{4}C\,C'^2\sin(2\epsilon'+\epsilon)\sin 7\omega t$$

$$+\frac{\lambda}{4}C'^3\cos 3\epsilon'\cos 9\omega t+\frac{\lambda}{4}C'^3\sin 3\epsilon'\,\mathrm{sim}\,9\omega t \qquad (9)$$

The values of C, C', ϵ, ϵ' are then obtained from the condition that the following intergral be a minimum

$$F(C,C',\epsilon,\epsilon')=2\int_0 E^2(t)\,dt$$

$$=1-\frac{3\lambda}{2}C^2C'\cos(\epsilon'-2\epsilon)+(2\lambda C^3C'(1-3\omega^2)$$

$$+\frac{15}{4}\lambda^2C^3C'^3+\frac{15}{8}\lambda^2C^5C')\cos(\epsilon'-3\epsilon)$$

$$-2\left[1-\omega^2+\frac{3\lambda}{4}(C^2+2C'^2)\right]C\cos\epsilon$$

$$-2\alpha\omega C\sin\epsilon+(\alpha^2\omega^2+(1-\omega^2)^2C^2+\frac{3\lambda}{2}(1-\omega^2)C^4$$

$$+\frac{5}{8}\lambda^2C^6+((1-9\omega^2)^2+9\alpha^2\omega^2)C'^2+6\lambda(1-5\omega^2)C^2C'^2$$

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$$+\frac{3\lambda}{2}(1-9\omega^2)C'^4+\frac{45}{8}\lambda^2 C^4C'^2+\frac{45}{8}\lambda^2 C^2 C'+\frac{5}{8}\lambda^2 C'^6 \quad (10)$$

We first let C, C', ϵ fixed and the value of ϵ' for which $F(C, C', \epsilon, \epsilon')$ is minimum, can easily be shown to be

$$\tan\epsilon' = \frac{\frac{3}{2}\sin 2\epsilon + C(6\omega^2-2-\frac{15}{8}\lambda C^2-\frac{15}{4}\lambda C'^2)\sin 3\epsilon}{\frac{3}{2}\cos 2\epsilon + C(6\omega^2-2-\frac{15}{8}\lambda C^2-\frac{15}{4}\lambda C'^2)\cos 3\epsilon} \quad (11)$$

Substituting for ϵ' from this equation into Eq. (10), we get a function $F_1(C, C', \epsilon)$ of three parameters C, C', ϵ

$$F_1(C,C',\epsilon) = -\lambda C' C^2 \sqrt{\frac{C^2(6\omega^2-2-\frac{15}{8}\lambda C^2-\frac{15}{4}\lambda C'^2)^2}{+\frac{9}{4}+3C(6\omega^2-2-\frac{15}{8}\lambda C^2-\frac{15}{4}\lambda C'^2)\cos\epsilon}}$$

$$-2\alpha\omega C\sin\epsilon-2(1-\omega^2+\frac{3\lambda}{4}C^2+\frac{3\lambda}{2}C'^2)C\cos\epsilon+$$

$$+1+((1-\omega^2)^2+\alpha^2\omega^2)C^2+((1-9\omega^2)^2+9\alpha^2\omega^2)C'^2+$$

$$+\frac{3\lambda}{2}(1-\omega^2)C^4+\frac{5}{8}\lambda^2C^6+6\lambda(1-5\omega^2)C^2C'^2$$

$$+\frac{3\lambda}{2}(1-9\omega^2)C'^4+\frac{45}{8}\lambda^2C^4C'^2+\frac{45}{8}\lambda^2C^2C'^4+\frac{5}{8}\lambda^2C'^6 \quad (12)$$

The value of ϵ, on fixing C, C', which renders $F_1(C,C',\epsilon)$ a mimimum can to a great accuracy be obtained in the following way. Let ϵ_0 be given by

$$\tan \epsilon_o = \frac{\alpha\omega}{(1-\omega^2 + \frac{3\lambda}{4} C^2 + \frac{3\lambda}{2} C'^2)} \quad (13)$$

This value clearly renders the second and third terms of Eq. (12) minimum . The right ϵ can then be to great accuracy be given by

$$\epsilon = \epsilon_o - \frac{\left(\frac{\partial F_1}{\partial \epsilon}\right)_{\epsilon=\epsilon_o}}{\left(\frac{\partial^2 F_1}{\partial \epsilon^2}\right)_{\epsilon=\epsilon_o}} \quad (14)$$

We now to facilitate the expressions, use the following definitions

$$A = (6\omega^2 - 2 - \frac{15}{8} \lambda C^2 - \frac{15}{4} \lambda C'^2) C \quad (15)$$

$$Z = \sqrt{\alpha^2 \omega^2 + (1-\omega^2 + \frac{3}{4} \lambda C^2 + \frac{3\lambda}{2} C'^2)^2}$$

we then find

$$\left(\frac{\partial F_1}{\partial \epsilon}\right)_{\epsilon=\epsilon_o} = \frac{3/2 \lambda C'C^2 A \sin \epsilon_o}{\sqrt{A^2 + 3A \cos \epsilon_o + 9/4}}$$

$$\left(\frac{\partial^2 F_1}{\partial \epsilon^2}\right)_{\epsilon=\epsilon_o} = 2 C Z + \frac{\frac{3}{2} \lambda C' C^2 A \cos \epsilon_o}{\sqrt{A^2 + 3A \cos \epsilon_o + \frac{9}{4}}} +$$

$$+ \frac{\frac{9}{4} \lambda C' C^2 A^2 \sin^2 \epsilon_o}{(A^2 + 3A \cos \epsilon_o + \frac{9}{4})^{3/2}} \quad (16)$$

Now expanding $F_1(C,C',\epsilon)$ in powers of $\epsilon - \epsilon_o$ and substituting for ϵ given by Eq. (14) we get

$$F_2(C,C') = F_1(C,C',\epsilon_o) - 1/2 \frac{\left(\frac{\partial F_1}{\partial \epsilon}\right)^2_{\epsilon=\epsilon_o}}{\left(\frac{\partial^2 F_1}{\partial \epsilon^2}\right)_{\epsilon=\epsilon_o}} \tag{17}$$

It will be found that the value of C' that renders $F_2(C,C')$ a minimum is quite small , then we can expand the function $F_2(C,C')$ up to second power in C' . The value of C' that renders $F_2(C,C')$ a minimum can then be easily found. The expansion of $F_2(C,C')$ can be expressed

$$\begin{aligned} F_2(C,C') = {} & 1+((1-\omega^2)^2+\alpha^2\omega^2)C^2+\frac{3\lambda}{2}(1-\omega^2)C^4+\frac{5}{8}\lambda^2 C^6 - 2CZ_o \\ & - C'C^2\sqrt{A_o^2+3A_o\cos\epsilon_{oo}+\frac{9}{4}} + [(1-9\omega^2)^2+9\alpha^2\omega^2+6\lambda(1-5\omega^2)C^2 \\ & + \frac{45}{8}\lambda^2 C^4 - 3\lambda C\cos\epsilon_{oo} - \frac{9}{16}\frac{\lambda^2 C^3\sin^2\epsilon_{oo}A_o^2}{(A_o^2+3A_o\cos\epsilon_{oo}+\frac{9}{4})Z_o}]C'^2 \end{aligned} \tag{18}$$

where A. , Z. are as given by Eq. (15) on putting C'=0 and .. as given by Eq. (13), on putting C'=0

The value of C' that renders $F_2(C,C')$ a minimum (fixing C) is given by

$$C' = \frac{\lambda C^2\sqrt{A_o^2+3A_o\cos\epsilon_{oo}+9/4}}{2\left[(1-9\omega^2)^2+9\alpha^2\omega^2+6\lambda(1-5\omega^2)C^2+\frac{45}{8}\lambda^2C^4-3\lambda C\cos\epsilon_{oo} - \frac{9}{16}\frac{\lambda^2C^3A_o^2\sin^2\epsilon_{oo}}{Z_o(A_o^2+3A_o\cos\epsilon_{oo}+\frac{9}{4})}\right]} \tag{19}$$

Substituting this value in Eq. (18), we finally find a function of only one parameter C

$$F_3(C) = 1+[(1-\omega^2)+\alpha^2\omega^2]C + \frac{3\lambda}{2}(1-\omega^2)C + \frac{5}{8}\lambda^2 C - 2CZ_o$$

$$-\frac{\lambda^2 C^4}{2}\;\frac{(A_o^2 + 3A_o\cos\epsilon_{oo} + \frac{9}{4})}{[(1-9\omega^2)^2 + 9\alpha^2\omega^2 + 6\lambda(1-5\omega^2)C^2 + \frac{45}{8}\lambda^2 C^4 - 3\lambda C\cos\epsilon_{oo}}$$

$$\overline{-\frac{9}{16}\;\frac{\lambda^2 C^3 A_o^2 \sin^2\epsilon_{oo}}{Z_o(A_o^2 + 3A_o\cos\epsilon_{oo} + \frac{9}{4}}}$$

The value of C which renders F_3 (C) aminimum can then only be done numerically .

The value of ω^2 is chosen such that the average power absorbed from the external force be maximum. The average power absorbed is given by Eq. (3).

<u>Numerical Procedure</u>:

The value of α is given the numerical value

$$\alpha = 0.002$$

as in the practical case of an ideal resonating system. In such a case it is found that for $\lambda = 0$,

$$\epsilon = \frac{\pi}{2} \quad , \quad C = 500 \quad , \quad P = 250 \; .$$

The values of λ are then increased by small amounts $\lambda = 10^{-9}$, ... In each case a numerical computer program is set to compute the value of C (for given ω^2) for which F_3 (C) given by Eq. (20) is minimum.

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It may be note that in such a procedure there is always two minima for the function $F_3(C)$ and one must be careful to choose the value of C corresponding to least value of F_3 (C) (best solution).

The value of ω^2 is now changed until one finds the requried resonant value of ω^2 which renders the power absorbed P a miximum.

The whole procedure is repeated for different values of λ and in each case the resonant value of ω^2, and then the corresponding solution values of C, C', ϵ, ϵ' and the power P are computed. In table (1) the results of these computations are illustrated. In Figures 1, 2, 3, 4, 5 are shown the behaviour of the variations of C, ϵ, C', ϵ', P as functions of λ.

Conclusion :

The results illustrated clearly show that as the value of λ is increased to the small numerical value $\lambda = 10^{-7}$ the power absorbed drops from the ideal value ($\lambda = 0$) P =250 to the small value P = 47.18 . This means that for such minute values of λ the oscillating system cannot be used as a resonant system that can absorb a considerable power from the external field. It is known that at resonance the amplitude is very big and in such a case the effect of anharmonic terms (terms corresponding to λx^3 in the eguation) is important. This means that we must be very careful in practical application either in mechanical or electrical resonating systems to choose the value of λ to be very small not exceeding

$$\lambda = 2 \times 10^{-8}$$

REFERENCES

1- Hayashi , C. , Non-linear oscillations in physical systems, Mc. Graw-Hill , New York.

2- Hale , J. , Oscillationssin non- linear systems. Mc. Graw-Hill, New York (1963) .

3- Kryloff , K. , and Bogoliubov , B., Introduction to non- linear mechanics Princeton Univ. Press. (1947) .

4- Cesari , L. , and Hales , J. K. , Proc. Am. Math. Soc. vol. 8 (1957).

5- Gambill , R . A . , and Hale , J. K., J. Rational Mech. Anal. vol. 5 No. 2 (1956).

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Figure Caption

Fig. 1 - Variation of the a mplitude C versus the anharmonic parameter λ .

Fig. 2 - Variation of the phase difference ϵ versus λ .

Fig. 3 - Variation of the amplitude C' versus λ .

Fig. 4 - Variation of the phase difference ϵ' versus λ .

Fig. 5 - Variation of the phase a bsorbed P versus λ .

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Fig (1)

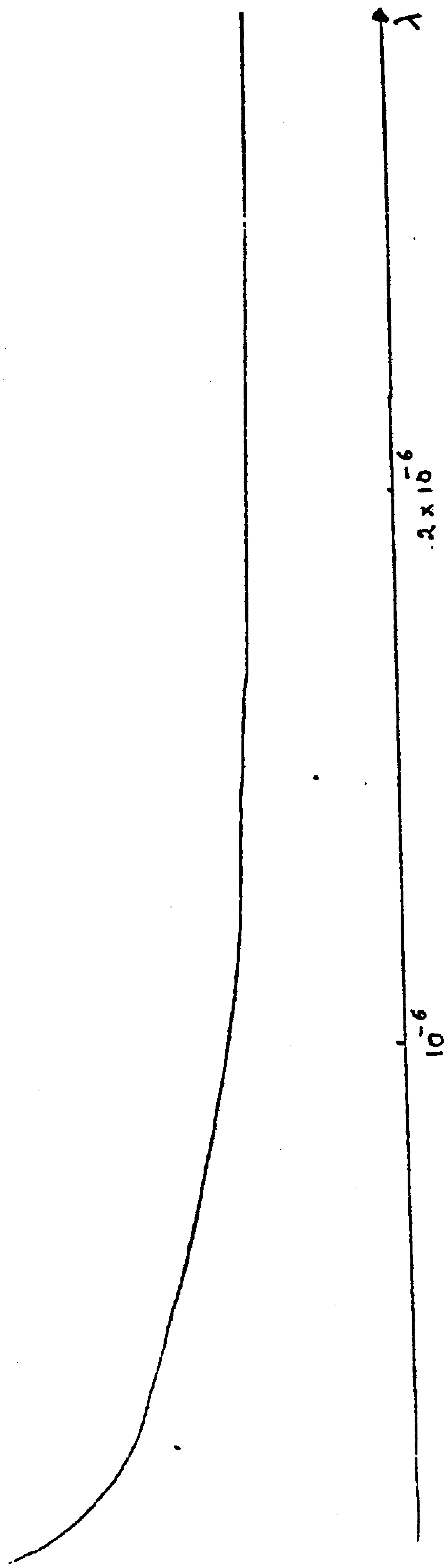

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Fig.(2)

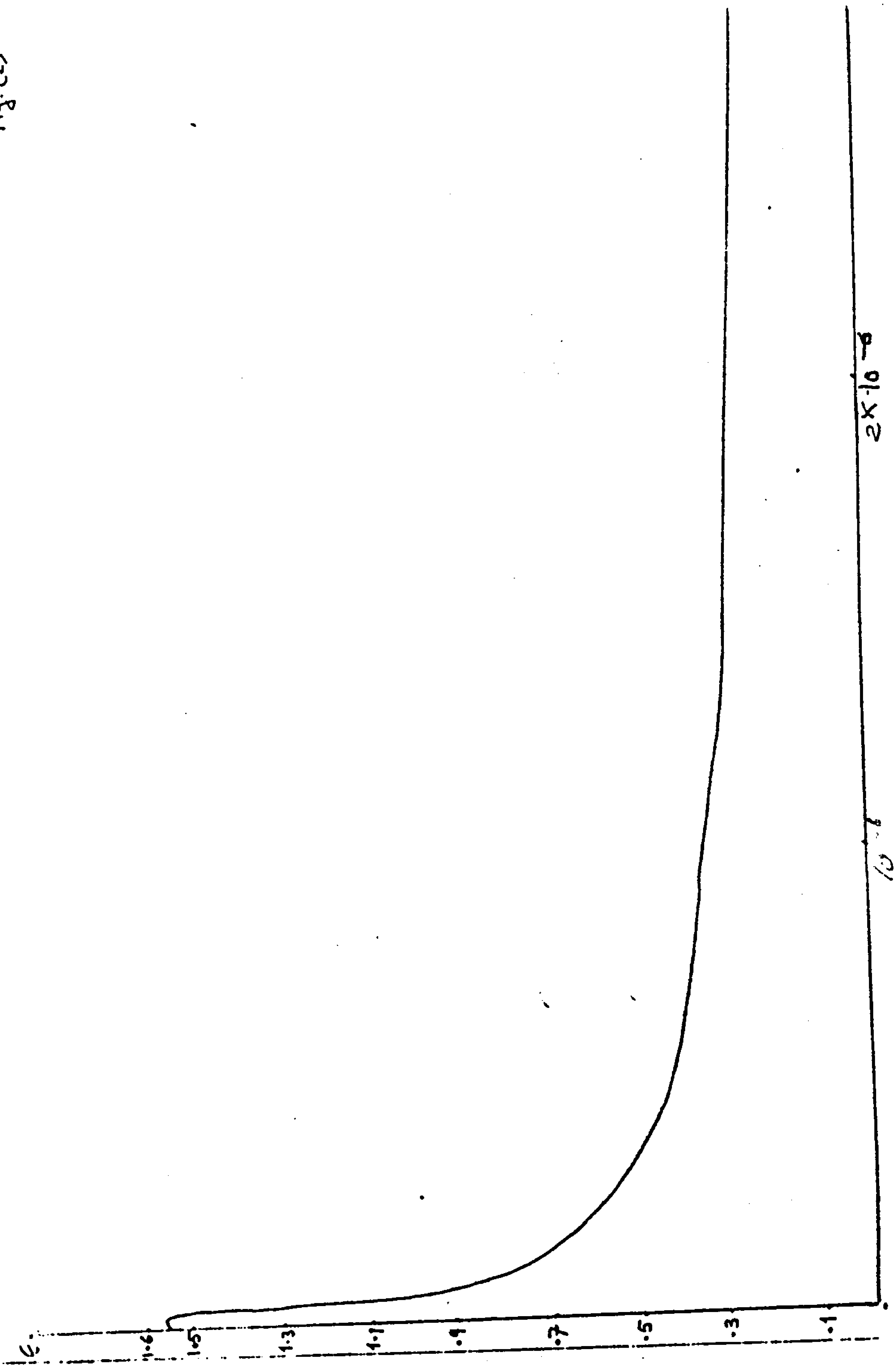

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Fig.(3)

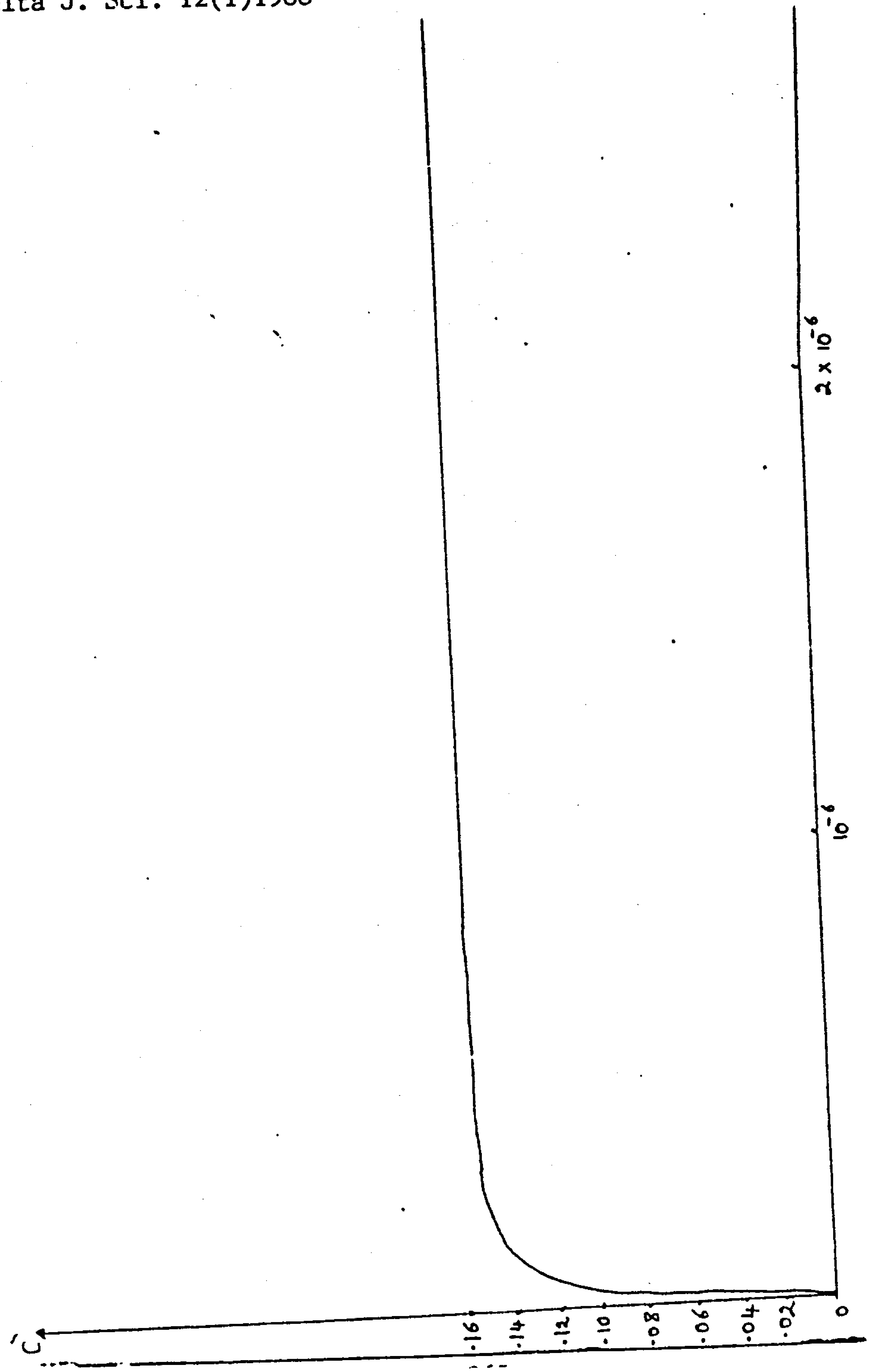

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Fig.(4)

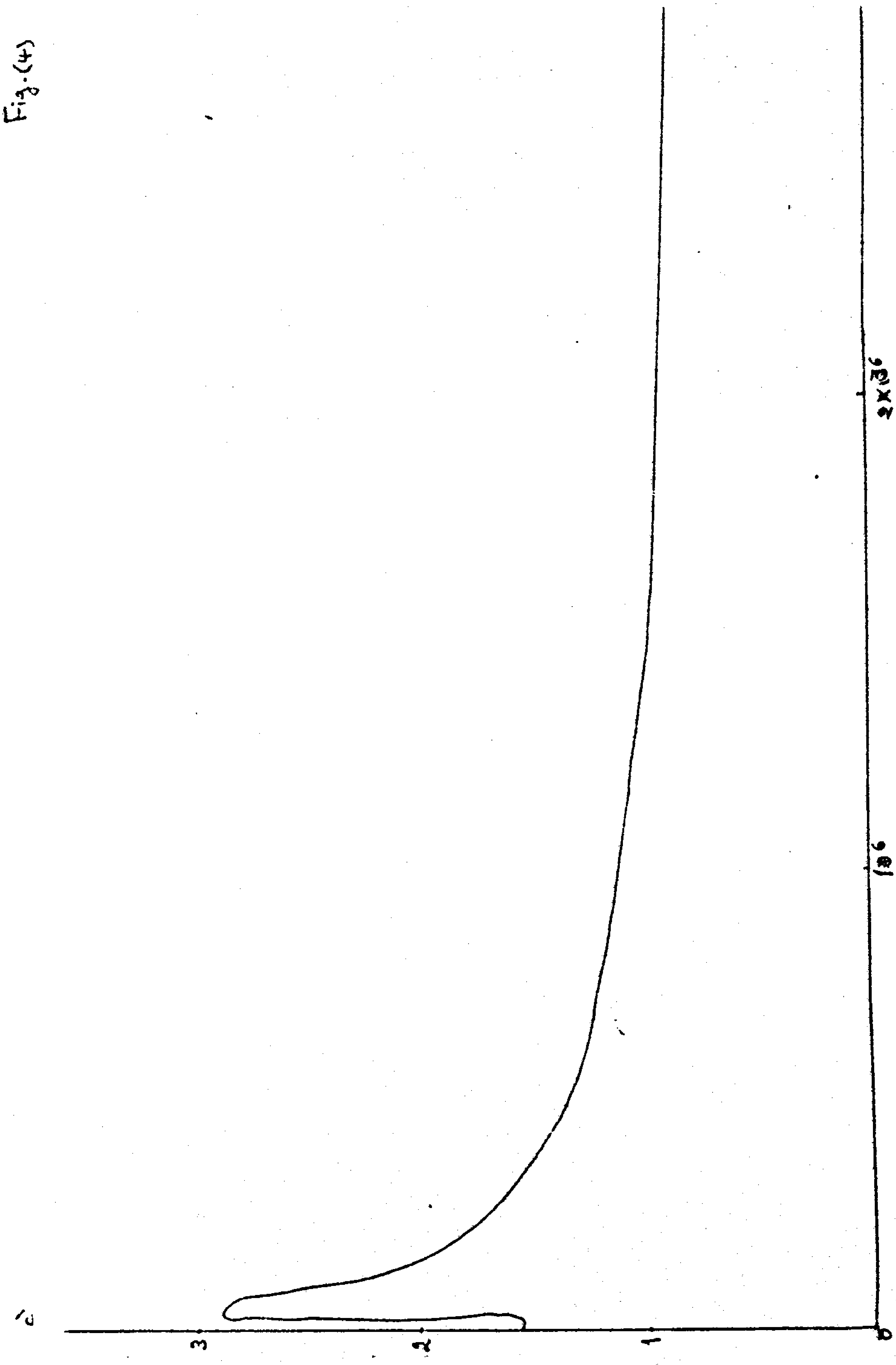

Fig. (5)

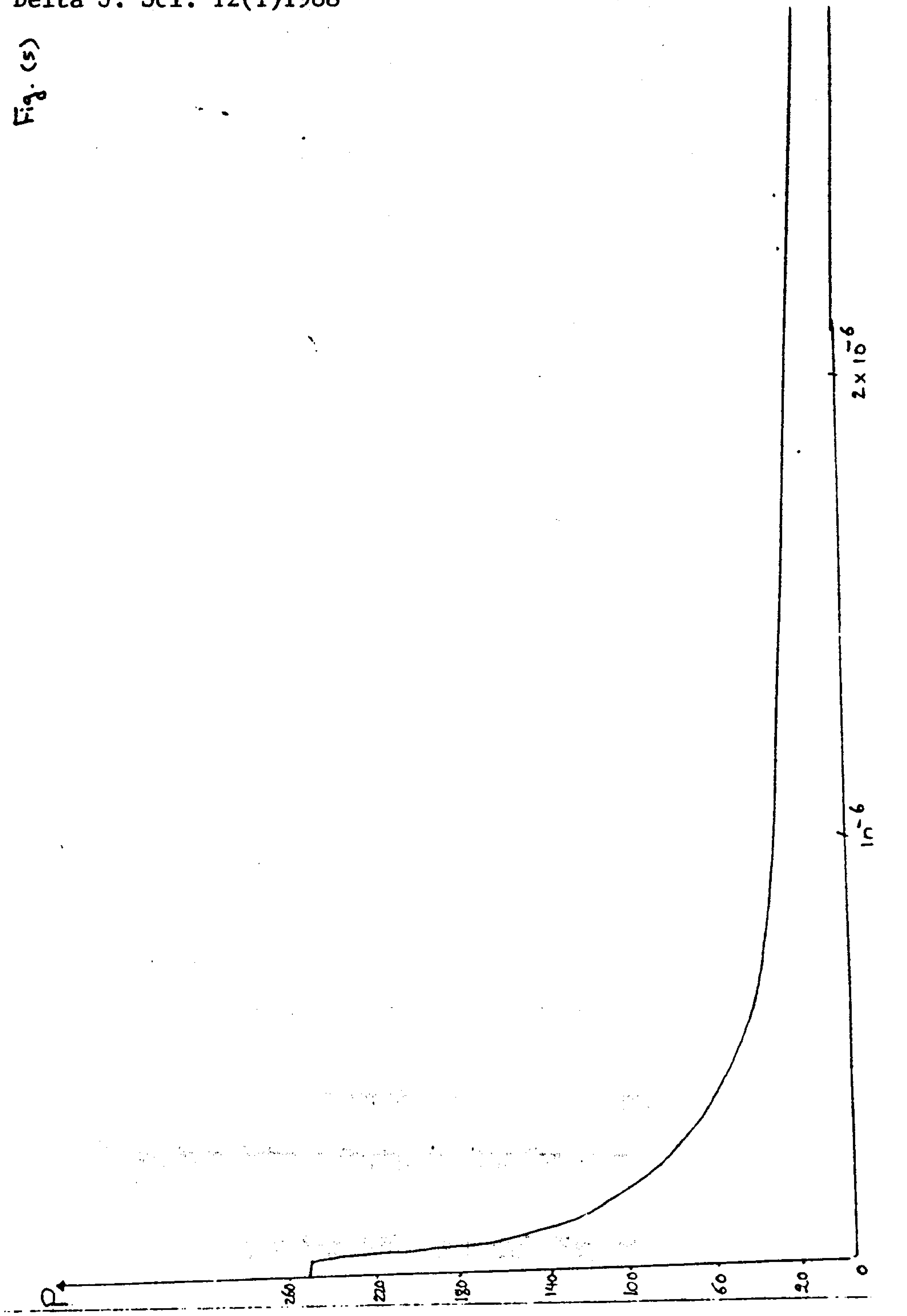

Table (1)

[illegible]	[illegible]	C	P	C'	ε	ε'	P
0	1	500	0	0	1.57070632	1.57154633	250
10^{-9}	1.0002	499.94	0	.00390415	1.5770674	1.552733096	249.99008
10^{-8}	1.0019	499.47	$5.332\ 10^{-9}$	.0388735	1.5851953	1.52834893	249.946228
$2\ 10^{-8}$	1.0037	499.00	$8.355\ 10^{-8}$	.07740722	1.553033747	1.62483303	249.9217173
$3\ 10^{-8}$	1.0046	479.53	$2.84392\ 10^{-7}$	.10297596	1.29147175	2.40951885	231.0016363
$4\ 10^{-8}$	1.0052	452.11	$4.97922\ 10^{-7}$	.115019419	1.135026635	2.8788541	205.4612493
$6\ 10^{-8}$	1.0062	410.43	$9.566\ 10^{-7}$	.12897854	.96732467	2.90122551	169.4909476
10^{-7}	1.0075	355.32	$1.74041\ 10^{-6}$	.1393413	.7941539	2.3817135	127.1940093
$5\ 10^{-7}$	1.0133	215.79	$7.35366\ 10^{-6}$	.15535804	.44936087	1.34733566	47.17914924
10^{-6}	1.0169	172.49	$1.2426\ 10^{-5}$	.1582223	.3554825	1.0652345	30.267775
$3\ 10^{-6}$	1.0245	120.23	$2.68804\ 10^{-5}$	.15982502	.24584953	.73680395	14.8089554
10^{-5}	1.0367	80.69	$6.030194\ 10^{-5}$	.159530741	.16501804	.494312291	6.7479967
10^{-4}	1.0788	37.42	$2.57837\ 10^{-4}$	.154102104	.07778687	.23262771	1.5101236

تـأثير الحدود الغير توافقيـة على رنين نظام متذبذب

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يتعلق البحث بدراسة تأثير حدود لا توافقة فى نظام متذبذب تحت تأثير قوى نبذبة جبرية خارجية خصوصا فى حالة الرنين عند ما تكون الطاقة الممتصة (أو المفقودة) من القوة الخارجية أكبر ما يمكن .

عند اهمال القوة اللا توافقية فأن نظام المتذبذب التوافقى تحت تأثير قوى مجهرة يحكم بالمعادلة (باستخدام وحدات لا بعدية)

$$\ddot{x} + x + \alpha \dot{x} = \cos \omega t$$

و من المعلوم أن حل هذه المعادلة هو

$$x = \frac{\cos(\omega t - \frac{\pi}{2})}{\sqrt{(1-\omega^2)^2 + \alpha^2 \omega^2}}$$

وتتعين الطاقة الممتصة بالقوة الخارجية من العلاقة

$$P = \frac{1}{2\sqrt{\alpha^2 + (\frac{1-\omega^2}{\omega})^2}}$$

وتكون هذه نهاية عظمى عندما $\omega^2 = 1$

وفى حالة وجود حد لا توافقى وهو موضوع هذا البحث تصبح المعادلة (باستخدام وحدات لا بعدية)

$$\ddot{x} + x + \alpha \dot{x} + \lambda x^2 = P \cos \omega t$$

وقد وجد أنه لقيم λ التى تتراوح بين الصفر ، 2×10^{-8} فأن السلوك يكون كما فى حالة $\lambda = 0$ أى أن النظام المتذبذب يكون له نفس القدرة على الامتصاص من المجال الخارجى فى حالة الرنين .

ولقيم λ الاكبر بالرغم من صغرها عدديا فان تاثير الحدود الغير توافقية يكون كبيرا بحيث ينعدم تقريبا اعتبار المجموعة مجموعة رنين يكون امتصاص الطاقة عندها كبيرا .